Chiral Lagrangian

— A Beautiful Application of Effective Field Theory

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EFT in a nutshell

Nature has many scales.

Science progresses as we can treat one scale at a time.

Coarse-graining over short-distance (high-energy) scales leads to an effective field theory (EFT) at long distances (low energies).

Even if we do not know the detail of the full theory, we can parameterize our ignorance in an EFT.

Some typical EFTs:

- Why is the sky blue?
  Rayleigh scattering cross section $\sigma \propto E_\gamma^4 \quad \Rightarrow \quad$ blue light dominant
- Fermi's theory of the weak interactions
- Chiral Lagrangian in QCD

$$\mathcal{L} = \frac{f^2}{4}\mathrm{tr}\left[(\partial_\mu \Sigma^\dagger)(\partial^\mu \Sigma)\right] + \cdots$$

- Techincolor and composite Higgs theories
- Dimension 5 operator giving neutrino masses

$$\mathcal{L} \supset \frac{1}{\Lambda}(LH)(LH)$$

- Dimension 6 operator leading to proton decay

$$\mathcal{L} \supset \frac{1}{\Lambda^2}QQQL$$

Some of my publications related to EFTs:

Electroweak chiral Lagrangian:

- Phys.Rev.D 77 (2008) 055003
- Eur.Phys.J.C 67 (2010) 51-56

Dark matter (DM) effective field theory:

- Nucl.Phys.B 854 (2012) 350-374 [90 citations]
- Nucl.Phys.B 860 (2012) 115-151 [71 citations]

Composite Higgs & DM production mechanism:

- **Phys.Rev.Lett.** 125 (2020) 2, 021801
- JHEP 01 (2019) 130
- Phys.Rev.D 102 (2020) 7, 075018

Minimal and simplified DM:

- Phys.Rev.D 92 (2015) 11, 115004
- Nucl.Phys.B 921 (2017) 181-210

Chiral symmetry in QCD

Consider the QCD Lagrangian, keeping only the three lightest quarks, *u*, *d* and *s*:

$$\mathcal{L} = \sum_{i=1}^{3} (\bar{q}_i i \not{D} q_i - m_i \bar{q}_i q_i) - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

where $D_\mu = \partial_\mu - igT^a A^a_\mu$ is the covariant derivative, $T^a = \lambda^a/2$ is $SU(3)$ generators in the 3 representation.

Note that the kinetic term can be written as

$$\sum_{i=1}^{3} \bar{q}_i i \not{D} q_i = \sum_{i=1}^{3} (\bar{q}_{Li} i \not{D} q_{Li} + \bar{q}_{Ri} i \not{D} q_{Ri})$$

which respects a $U(3)_L \times U(3)_R$ symmetry.

One combination of these transformations, the $U(1)_A$ transformation where $q_i \to e^{i\alpha\gamma_5} q_i$ is in fact not a symmetry of the quantum theory, due to anomalies.

This leaves us with a $U(1)_V \times SU(3)_L \times SU(3)_R$ symmetry.

The $U(1)_V$ is just baryon number, under which both left- and right-handed quarks of all flavors pick up a common phase.

The remaining $SU(3)_L \times SU(3)_R$ symmetry is called chiral symmetry.

However, $SU(3)_L \times SU(3)_R$ is not an exact symmetry of QCD.

The mass term couples left- and right-handed quarks, so it is not invariant under the chiral symmetry.

$$\sum_i m_i \bar{q}_i q_i = \sum_{i,j} \bar{q}_{Ri} M_{ij} q_{Lj} + \mathrm{h.c.}$$

$$M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

If the mass matrix M were a dynamical field, transforming under $SU(3)_L \times SU(3)_R$ as

$$M \to RML^\dagger$$

then the Lagrangian would be chirally invariant.

Chiral symmetry may be spontaneously broken due to a field M.

This is called treating M as a spurion.

The chiral symmetry is broken to the extent that $M \neq RML^\dagger$.

Since m_u and m_d are much smaller than m_s, $SU(2)_L \times SU(2)_R$ is not broken as badly as $SU(3)_L \times SU(3)_R$.

♣ If all three quark masses were equal but nonzero, then QCD would respect an $SU(3)_V \subset SU(3)_L \times SU(3)_R$ symmetry, where one sets $L = R$.

This is the $SU(3)$ symmetry of Gell-Mann.

♣ Since $m_d - m_u$ is small, $SU(2)_V \subset SU(3)_V$, where $L = R$ and they act nontrivially only on the u and d quarks, is quite a good symmetry.

It is known as isospin symmetry.

Independent vectorlike phase rotations of the three flavors of quarks

$$q_i \to e^{i\alpha_i} q_i$$

are exact symmetries.

These three $U(1)$ symmetries are linear combinations of the baryon number B, the 3rd component of the isospin symmetry I_3, and the hypercharge Y.

The latter two are violated by the weak interaction, but not by the strong or electromagnetic forces.

This is not the whole story.

The QCD vacuum spontaneously breaks the chiral $SU(3)_L \times SU(3)_R$ symmetry down to Gell-Mann's $SU(3)_V$ via the quark condensate:

$$\langle 0|\bar{q}_{Rj} q_{Li}|0\rangle = \Lambda^3 \delta_{ij}$$

which transforms as a $(3, \bar{3})$ under $SU(3)_L \times SU(3)_R$.

Here Λ has a dimension of mass.

If one redefines the quark fields by a chiral transformation, then the Kronecker δ-function above gets replaced by a general $SU(3)$ matrix $\Sigma = LR^{\dagger}$,

$$
\begin{aligned}
\langle 0|\bar{q}_{Rj} q_{Li}|0\rangle &\rightarrow L_{im}\langle 0|\bar{q}_{Rn} q_{Lm}|0\rangle R^{\dagger}_{nj} \\
&= L_{im}\Lambda^3 \delta_{mn} R^{\dagger}_{nj} \\
&= \Lambda^3 (LR^{\dagger})_{ij} \\
&\equiv \Lambda^3 \Sigma_{ij}
\end{aligned}
$$

If $L = R$ (a Gell-Mann's $SU(3)_V$ transformation), then $\Sigma_{ij} = \delta_{ij}$ which shows that the condensate leaves unbroken the $SU(3)_V$ symmetry.

For $L \neq R$, Σ_{ij} represents a different vacuum from δ_{ij}.

If it wasn't for the explicit breaking of $SU(3)_L \times SU(3)_R$ by quark masses in the QCD Lagrangian, these two different vacua would be degenerate.

By Goldstone's theorem, there would be eight exact Goldstone bosons — one for each of the eight broken generators.

We parameterize these Goldstone boson excitations by replacing

$$\Sigma \to \Sigma(x) \equiv \exp\left[\frac{2i}{f}\pi(x)\right]$$

$$\pi(x) = T^a \pi^a(x)$$

where T^a are the $SU(3)$ generators ($a = 1, \ldots, 8$) in the defining representation normalized to $\mathrm{tr}(T^a T^b) = (1/2)\delta^{ab}$.

f is a parameter with dimension of mass which we will relate to the pion decay constant f_π, and $\pi^a(x)$ are eight mesons transforming as an octet under $SU(3)_V$.

The interaction of these mesons are represented by the chiral Lagrangian.

The transformation of $\Sigma(x)$ under $SU(3)_L \times SU(3)_R$ is given by

$$\Sigma(x) \to L\Sigma(x)R^\dagger$$

The chiral Lagrangian is required to be invariant under this transformation.

Note that $SU(3)_L \times SU(3)_R$ is not exact.

It is broken by nonzero quark masses and by the electric charges of the quarks.

We will have to incorporate those effects.

Quantum numbers of the meson octet

It is useful to use the basis for $SU(3)$ generators $T^a = (1/2)\lambda^a$, where λ^a are Gell-Mann matrices:

$$\lambda^i = \begin{pmatrix} \sigma_i & \\ & 0 \end{pmatrix} \quad (i = 1, 2, 3)$$

$$\lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix},$$

$$\lambda^6 = \begin{pmatrix} 0 & \\ & \sigma_1 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & \\ & \sigma_2 \end{pmatrix},$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

For $N = 2$, the pion matrix is

$$\begin{aligned} \phi \equiv \sum_{a=1}^{3} \phi_a \sigma^a &= \phi_1 \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} + \phi_2 \begin{pmatrix} & -i \\ i & \end{pmatrix} + \phi_3 \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\ &= \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \end{aligned}$$

where we have defined

$$\begin{aligned} &\pi^0 \equiv \phi_3 \\ &\pi^+ \equiv \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \\ \Leftrightarrow \quad &\pi^- \equiv (\pi^+)^\dagger = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \end{aligned}$$

Why does $\pi^+ \equiv \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$ have the electric charge $+1$?

♣ Method 1: By comparing

$$\phi_{ij} \propto \langle \bar{q}_{Rj} q_{Li} \rangle = \begin{pmatrix} \langle \bar{u}_R u_L \rangle & \langle \bar{d}_R u_L \rangle \\ \langle \bar{u}_R d_L \rangle & \langle \bar{d}_R d_L \rangle \end{pmatrix}$$

$$\phi_{ij} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$$

we get $\pi^+ \sim \langle \bar{d} u \rangle$. Then, from the electric charges of the quarks

$$Q(u) = \frac{2}{3}, \quad Q(d) = -\frac{1}{3}$$

we see that the electric charge of π^+ is

$$Q(\pi^+) = Q(\langle \bar{d} u \rangle) = -\left(-\frac{1}{3}\right) + \frac{2}{3} = +1$$

♣ Method 2: The generator of the electric charge is

$$Q = T_{3L} + Y, \quad Y = T_{3R} + \frac{B}{2} \quad \Leftrightarrow \quad Q = T_{3L} + T_{3R} + \frac{B}{2}$$

Under $U(1)_{em}$, the left- and right-handed quarks transform as

$$q_L \to e^{i(T_{3L}+\frac{B}{2})\theta} q_L, \quad q_R \to e^{i(T_{3R}+\frac{B}{2})\theta} q_R$$

Thus, $\phi \sim \langle q_L \bar{q}_R \rangle$ transforms as

$$\begin{aligned}
\phi \quad \to \quad & e^{i(\frac{\sigma_3}{2}+\frac{B}{2})\theta}\, \phi\, e^{-i(\frac{\sigma_3}{2}+\frac{B}{2})\theta} = e^{i\frac{\sigma_3}{2}\theta}\, \phi\, e^{-i\frac{\sigma_3}{2}\theta} \\
& = \begin{pmatrix} e^{i\frac{1}{2}\theta} & \\ & e^{-i\frac{1}{2}\theta} \end{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} e^{-i\frac{1}{2}\theta} & \\ & e^{i\frac{1}{2}\theta} \end{pmatrix} \\
& = \begin{pmatrix} * & e^{i\theta} * \\ e^{-i\theta} * & * \end{pmatrix}
\end{aligned}$$

that is, the upper-right element of ϕ has electric charge of $+1$.

For $N = 3$, the meson matrix is

$$\phi = \sum_{a=1}^{8} \phi_a \lambda^a = \begin{pmatrix} \phi_3 + \frac{\phi_8}{\sqrt{3}} & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{\phi_8}{\sqrt{3}} & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}$$

$$= \begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

where we have defined

$$\pi^0 \equiv \phi_3, \quad \eta \equiv \phi_8$$

$$\pi^+ \equiv \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \quad \Rightarrow \quad \pi^- \equiv (\pi^+)^\dagger = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$K^+ \equiv \frac{1}{\sqrt{2}}(\phi_4 - i\phi_5) \quad \Rightarrow \quad K^- \equiv (K^+)^\dagger = \frac{1}{\sqrt{2}}(\phi_4 + i\phi_5)$$

$$K^0 \equiv \frac{1}{\sqrt{2}}(\phi_6 - i\phi_7) \quad \Rightarrow \quad \bar{K}^0 \equiv (K^0)^\dagger = \frac{1}{\sqrt{2}}(\phi_6 + i\phi_7)$$

Why do $\pi^0, \eta, \pi^+, K^+, K^0$ have the assigned electric charges?

♣ The transformation of

$$\phi = \sum_{a=1}^{8} \phi_a \lambda^a = \begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

under $U(1)_{em}$ is

$$\begin{aligned} \phi \quad \to \quad & \begin{pmatrix} e^{i\frac{1}{2}\theta} & & \\ & e^{-i\frac{1}{2}\theta} & \\ & & e^{-i\frac{1}{2}\theta} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} e^{-i\frac{1}{2}\theta} & & \\ & e^{i\frac{1}{2}\theta} & \\ & & e^{i\frac{1}{2}\theta} \end{pmatrix} \\ = & \begin{pmatrix} * & e^{i\theta}* & e^{i\theta}* \\ e^{-i\theta}* & * & * \\ e^{-i\theta}* & * & * \end{pmatrix} \end{aligned}$$

The meson matrix can be written in the two conventions as

$$\pi(x) \equiv \sum_{a=1}^{8} \phi_a \frac{\lambda^a}{2} = \frac{1}{2}\phi(x) \quad \Leftrightarrow \quad 2\pi(x) = \phi(x)$$

As a result,

$$\Sigma(x) \equiv \exp\left[\frac{2i}{f}\pi(x)\right] = \exp\left[\frac{i}{f}\phi(x)\right]$$

Under an $SU(3)_V$ transformation $L = R \equiv V$,

$$\begin{aligned}\Sigma \rightarrow\ & V\Sigma V^\dagger \\ & = \exp\left[\frac{2i}{f}V\pi(x)V^\dagger\right] = \exp\left[\frac{i}{f}V\phi(x)V^\dagger\right]\end{aligned}$$

$$\Rightarrow \quad \pi(x) \rightarrow V\pi(x)V^\dagger \quad \text{or} \quad \phi(x) \rightarrow V\phi(x)V^\dagger$$

It is easy to check that

$$\mathrm{tr}\left[\pi^2(x)\right] = \frac{1}{4}\mathrm{tr}\left[\phi^2(x)\right] = \sum_{a,b}\phi_a\phi_b \mathrm{tr}\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right)$$

$$= \sum_{a,b}\phi_a\phi_b\frac{1}{2}\delta^{ab} = \frac{1}{2}\sum_a \phi_a^2$$

$$= \frac{1}{2}(\pi^0)^2 + \frac{1}{2}\eta^2 + \pi^+\pi^- + K^+K^- + K^0\bar{K}^0$$

which is a manifestly $SU(3)_V$ invariant operator.

Proof for the last equality is simple: $\pi^0 \equiv \phi_3, \quad \eta \equiv \phi_8,$

$$\pi^+ \equiv \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \quad \Rightarrow \quad \pi^+\pi^- = \frac{1}{2}(\phi_1^2 + \phi_2^2)$$

$$K^+ \equiv \frac{1}{\sqrt{2}}(\phi_4 - i\phi_5) \quad \Rightarrow \quad K^+K^- = \frac{1}{2}(\phi_4^2 + \phi_5^2)$$

$$K^0 \equiv \frac{1}{\sqrt{2}}(\phi_6 - i\phi_7) \quad \Rightarrow \quad K^0\bar{K}^0 = \frac{1}{2}(\phi_6^2 + \phi_7^2)$$

Proof for

$$\mathrm{tr}\left[\pi^2(x)\right] = \frac{1}{2}(\pi^0)^2 + \frac{1}{2}\eta^2 + \pi^+\pi^- + K^+K^- + K^0\bar{K}^0$$

by *Mathematica*:

```
In[1]:= pifield = 1/2 ( pi0 + eta/Sqrt[3]      Sqrt[2] piplus      Sqrt[2] Kplus
                        Sqrt[2] piminus        -pi0 + eta/Sqrt[3]  Sqrt[2] Kzero
                        Sqrt[2] Kminus         Sqrt[2] Kzerobar    -2 eta/Sqrt[3] );

In[2]:= Tr[pifield.pifield] // Simplify

Out[2]= 1/2 (eta^2 + 2 Kminus Kplus + 2 Kzero Kzerobar + pi0^2 + 2 piminus piplus)
```

$$\mathrm{pifield} = \frac{1}{2}\begin{pmatrix} \mathrm{pi0} + \frac{\mathrm{eta}}{\sqrt{3}} & \sqrt{2}\,\mathrm{piplus} & \sqrt{2}\,\mathrm{Kplus} \\ \sqrt{2}\,\mathrm{piminus} & -\mathrm{pi0} + \frac{\mathrm{eta}}{\sqrt{3}} & \sqrt{2}\,\mathrm{Kzero} \\ \sqrt{2}\,\mathrm{Kminus} & \sqrt{2}\,\mathrm{Kzerobar} & -\frac{2\,\mathrm{eta}}{\sqrt{3}} \end{pmatrix};$$

$$\mathrm{Out[2]} = \frac{1}{2}\left(\mathrm{eta}^2 + 2\,\mathrm{Kminus}\,\mathrm{Kplus} + 2\,\mathrm{Kzero}\,\mathrm{Kzerobar} + \mathrm{pi0}^2 + 2\,\mathrm{piminus}\,\mathrm{piplus}\right)$$

Two basic principles of chiral Lagrangian

To write down the chiral Lagrangian, we are guided by two basic principles of effective field theory:

♣1. The chiral Lagrangian must be invariant under the chiral symmetry

$$\Sigma \to L\Sigma R^\dagger$$

We can incorporate symmetry breaking effects by including the quark mass matrix M, requiring that the chiral Lagrangian be invariant under the chiral symmetry if M were to transform as

$$M \to RML^\dagger$$

♣2. The other principle is that the chiral Lagrangian be an expansion of local operators suppressed by powers of momenta or meson masses divided by a cutoff Λ.

The cutoff is set by the scale of physics we are ignoring, such as the ρ, K^*, ω and η' mesons (with masses $m_\rho = 770$ MeV, $m_{K^*} = 892$ MeV, $m_\omega = 782$ MeV and $m_{\eta'} = 958$ MeV).

In practice, the cutoff seems to be at $\Lambda \simeq 1$ GeV in many processes. This cutoff is to be compared with

$$
\begin{aligned}
m_{\pi^\pm} &= 140 \text{ MeV}, \\
m_{K^\pm} &= 494 \text{ MeV}, \\
m_\eta &= 548 \text{ MeV}
\end{aligned}
$$

For purely mesonic processes, meson masses always appear squared, which helps.

The chiral perturbation works far better for pions than kaons or the η.

This is a reflection of the fact that $SU(2)_L \times SU(2)_R$ is a much better symmetry of QCD than $SU(3)_L \times SU(3)_R$.

The leading term

The lowest dimension chirally symmetric operator is

$$\begin{aligned}\mathcal{L}_0 &= \frac{f^2}{4}\operatorname{tr}\left[(\partial_\mu \Sigma^\dagger)(\partial^\mu \Sigma)\right] \\ &= \operatorname{tr}\left[(\partial_\mu \pi)(\partial^\mu \pi)\right] + \frac{1}{3f^2}\operatorname{tr}\left([\partial_\mu \pi, \pi][\partial^\mu \pi, \pi]\right) + \mathcal{O}\left(\frac{1}{f^3}\right)\end{aligned}$$

The $f^2/4$ prefactor is fixed by requiring that the mesons have canonically normalized kinetic terms:

$$\begin{aligned}\operatorname{tr}\left[(\partial_\mu \pi)(\partial^\mu \pi)\right] &= \frac{1}{2}(\partial \pi^0)^2 + \frac{1}{2}(\partial \eta)^2 + (\partial \pi^+)(\partial \pi^-) \\ &\quad + (\partial K^+)(\partial K^-) + (\partial K^0)(\partial \bar{K}^0)\end{aligned}$$

Proof for $\frac{f^2}{4}\mathrm{tr}\left[(\partial_\mu\Sigma^\dagger)(\partial^\mu\Sigma)\right] \supset \frac{1}{3f^2}\mathrm{tr}\left([\partial_\mu\pi,\pi][\partial^\mu\pi,\pi]\right)$

$$\Sigma(x) \equiv \exp\left[\frac{2i}{f}\pi(x)\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2i}{f}\pi\right)^n$$

$$= 1 + \frac{2i}{f}\pi(x) + \frac{1}{2!}\left(\frac{2i}{f}\pi\right)^2 + \frac{1}{3!}\left(\frac{2i}{f}\pi\right)^3 + \cdots$$

$$= 1 + \frac{2i}{f}\pi(x) - \frac{2}{f^2}\pi^2(x) - \frac{4i}{3f^3}\pi^3(x) + \cdots$$

$$\Sigma^\dagger(x) = \exp\left[-\frac{2i}{f}\pi(x)\right]$$

$$= 1 - \frac{2i}{f}\pi(x) - \frac{2}{f^2}\pi^2(x) + \frac{4i}{3f^3}\pi^3(x) + \cdots$$

$$\Rightarrow \quad \partial_\mu\Sigma = \frac{2i}{f}(\partial_\mu\pi) - \frac{2}{f^2}(\partial_\mu\pi^2) - \frac{4i}{3f^3}(\partial_\mu\pi^3) + \cdots$$

$$\partial_\mu\Sigma^\dagger = -\frac{2i}{f}(\partial_\mu\pi) - \frac{2}{f^2}(\partial_\mu\pi^2) + \frac{4i}{3f^3}(\partial_\mu\pi^3) + \cdots$$

$$\Rightarrow \quad \mathrm{tr}\left[(\partial\Sigma^\dagger)(\partial\Sigma)\right] = \mathrm{tr}\left\{ \left[\frac{2i}{f}(\partial\pi) - \frac{2}{f^2}(\partial\pi^2) - \frac{4i}{3f^3}(\partial\pi^3) + \cdots\right] \right.$$
$$\left. \times \left[-\frac{2i}{f}(\partial\pi) - \frac{2}{f^2}(\partial\pi^2) + \frac{4i}{3f^3}(\partial\pi^3) + \cdots\right] \right\}$$
$$= \frac{4}{f^2}\mathrm{tr}\left[(\partial\pi)(\partial\pi)\right] + \frac{4}{f^4}\mathrm{tr}\left[(\partial\pi^2)(\partial\pi^2)\right]$$
$$-2 \times \frac{8}{3f^4}\mathrm{tr}\left[(\partial\pi)(\partial\pi^3)\right]$$

$$\frac{4}{f^4}\mathrm{tr}\left[(\partial\pi^2)(\partial\pi^2)\right] = \frac{4}{f^4}\mathrm{tr}\left\{[\pi(\partial\pi) + (\partial\pi)\pi][\pi(\partial\pi) + (\partial\pi)\pi]\right\}$$
$$= \frac{8}{f^4}\left\{\mathrm{tr}\left[\pi(\partial\pi)\pi(\partial\pi)\right] + \mathrm{tr}\left[\pi^2(\partial\pi)^2\right]\right\}$$

$$-\frac{16}{3f^4}\mathrm{tr}\left[(\partial\pi)(\partial\pi^3)\right] = -\frac{16}{3f^4}\mathrm{tr}\left\{(\partial\pi)\left[(\partial\pi)\pi^2 + \pi(\partial\pi)\pi + \pi^2(\partial\pi)\right]\right\}$$
$$= -\frac{16}{3f^4}\left\{2\mathrm{tr}\left[(\partial\pi)^2\pi^2\right] + \mathrm{tr}\left[(\partial\pi)\pi(\partial\pi)\pi\right]\right\}$$

$$\Rightarrow \quad \mathrm{tr}\left[(\partial\Sigma^\dagger)(\partial\Sigma)\right] \supset \frac{4}{f^4}\mathrm{tr}\left[(\partial\pi^2)(\partial\pi^2)\right] - \frac{16}{3f^4}\mathrm{tr}\left[(\partial\pi)(\partial\pi^3)\right]$$

$$= \frac{1}{f^4}\left\{ \left(8 - \frac{16}{3}\right)\mathrm{tr}\left[(\partial\pi)\pi(\partial\pi)\pi\right] + \left(8 - \frac{32}{3}\right)\mathrm{tr}\left[(\partial\pi)^2\pi^2\right] \right\}$$

$$= \frac{8}{3f^4}\left\{\mathrm{tr}\left[(\partial\pi)\pi(\partial\pi)\pi\right] - \mathrm{tr}\left[(\partial\pi)^2\pi^2\right]\right\}$$

$$\Rightarrow \quad \frac{f^2}{4}\mathrm{tr}\left[(\partial\Sigma^\dagger)(\partial\Sigma)\right] \supset \frac{2}{3f^2}\left\{\mathrm{tr}\left[(\partial\pi)\pi(\partial\pi)\pi\right] - \mathrm{tr}\left[(\partial\pi)^2\pi^2\right]\right\}$$

On the other hand,

$$\frac{1}{3f^2}\mathrm{tr}\left([\partial_\mu\pi,\pi][\partial^\mu\pi,\pi]\right) = \frac{1}{3f^2}\mathrm{tr}\left\{[(\partial\pi)\pi - \pi(\partial\pi)][(\partial\pi)\pi - \pi(\partial\pi)]\right\}$$

$$= \frac{2}{3f^2}\left\{\mathrm{tr}\left[(\partial\pi)\pi(\partial\pi)\pi\right] - \mathrm{tr}\left[(\partial\pi)^2\pi^2\right]\right\} \quad \checkmark$$

Pion decay constant

From the above Lagrangian, it would seem that the only way to determine f is by looking at $\pi\pi$ scattering.

However there is a better way: by looking at the charged pion decay $\pi \to \mu\bar{\nu}_\mu$.

This occurs through the “semi-leptonic” weak interaction

$$\mathcal{L} \supset \frac{G_F}{\sqrt{2}} V_{ud}[\bar{u}\gamma^\mu(1-\gamma_5)d][\bar{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu] + \text{h.c.}$$

The matrix element of this operator sandwiched between $|\mu\bar{\nu}_\mu\rangle$ and $\langle\pi|$ factorizes, and the leptonic part is perturbative.

We are left with the nonperturbative part

$$\langle 0|\bar{u}\gamma^\mu(1-\gamma_5)d|\pi^-(p)\rangle \equiv i\sqrt{2}f_\pi p^\mu$$

The pion decay constant f_π is determined from the charged pion lifetime to be

$$f_\pi = 92.4 \pm 0.25 \text{ MeV}$$

Even if QCD is nonperturbative, we can easily match this charged current operator onto an operator in the chiral Lagrangian. This is because

$$\bar{u}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)d = j_L^{1\mu} + ij_L^{2\mu} \quad \Leftrightarrow \quad \bar{u}\gamma^\mu(1-\gamma_5)d = 2(j_L^{1\mu} + ij_L^{2\mu})$$

where $j_L^{a\mu}$ are the eight $SU(3)_L$ currents

$$j_L^{a\mu} \equiv \bar{q}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)T^a q$$

Proof for $\quad \bar{u}\gamma^{\mu}\left(\frac{1-\gamma_5}{2}\right)d = j_L^{1\mu} + i j_L^{2\mu}$:

$$T_1 = \frac{\lambda_1}{2} = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & \\ & 0 \end{pmatrix}, \quad T_2 = \frac{\lambda_2}{2} = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} & \\ & 0 \end{pmatrix}$$

$$\Rightarrow \quad T_1 + iT_2 = \begin{pmatrix} \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} & \\ & 0 \end{pmatrix}$$

$$\Rightarrow \quad j_L^{1\mu} + i j_L^{2\mu} = \bar{q}_L \gamma^{\mu}(T_1 + iT_2) q_L = \bar{u}_L \gamma^{\mu} d_L$$

Using Noether's theorem, we can compute the left-handed currents from the effective Lagrangian $\mathcal{L}_0$:

$$\mathcal{L}_0 = \frac{f^2}{4}\mathrm{tr}\left[(\partial\Sigma^\dagger)(\partial\Sigma)\right] = \frac{f^2}{4}(\partial\Sigma^\dagger_{ij})(\partial\Sigma_{ji})$$

$$\Rightarrow \quad \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\Sigma_{ij})} = \frac{f^2}{4}(\partial^\mu\Sigma^\dagger_{ji}), \quad \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\Sigma^\dagger_{ij})} = \frac{f^2}{4}(\partial^\mu\Sigma_{ji})$$

$\Rightarrow$ Noether's current is

$$\begin{aligned} j^\mu &\propto \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\Sigma_{ij})}\delta\Sigma_{ij} + \delta\Sigma^\dagger_{ij}\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\Sigma^\dagger_{ij})} \\ &= \frac{f^2}{4}(\partial^\mu\Sigma^\dagger_{ji})\delta\Sigma_{ij} + \delta\Sigma^\dagger_{ij}\frac{f^2}{4}(\partial^\mu\Sigma_{ji}) \\ &= \frac{f^2}{4}\mathrm{tr}\left[(\partial^\mu\Sigma^\dagger)\delta\Sigma + (\delta\Sigma^\dagger)(\partial^\mu\Sigma)\right] \end{aligned}$$

Under $SU(3)_L$ transformations,

$$\Sigma \to L\Sigma = e^{iT^a\theta^a}\Sigma \simeq (1 + iT^a\theta^a)\Sigma \quad \Rightarrow \quad \delta\Sigma = iT^a\theta^a\Sigma$$
$$\Sigma^\dagger \to \Sigma^\dagger L^\dagger = \Sigma^\dagger e^{-iT^a\theta^a} \simeq \Sigma^\dagger(1 - iT^a\theta^a) \quad \Rightarrow \quad \delta\Sigma^\dagger = -i\Sigma^\dagger T^a\theta^a$$

$\Rightarrow$ Noether's current for $SU(3)_L$ is

$$\begin{aligned}
j_L^\mu &\propto \frac{f^2}{4}\mathrm{tr}\left[(\partial^\mu\Sigma^\dagger)\delta\Sigma + (\delta\Sigma^\dagger)(\partial^\mu\Sigma)\right] \\
&= \frac{f^2}{4}\mathrm{tr}\left[(\partial^\mu\Sigma^\dagger)iT^a\theta^a\Sigma - i\Sigma^\dagger T^a\theta^a(\partial^\mu\Sigma)\right] \\
&= \frac{f^2}{4}i\theta^a\,\mathrm{tr}\left[(\partial^\mu\Sigma^\dagger)T^a\Sigma - \Sigma^\dagger T^a(\partial^\mu\Sigma)\right] \\
&= \frac{f^2}{4}i\theta^a\,\mathrm{tr}\left[-\Sigma^\dagger(\partial^\mu\Sigma)\Sigma^\dagger T^a\Sigma - \Sigma^\dagger T^a(\partial^\mu\Sigma)\right] \\
&= \frac{f^2}{4}i\theta^a\cdot(-2)\,\mathrm{tr}\left[\Sigma^\dagger T^a(\partial^\mu\Sigma)\right] \\
&= -i\frac{f^2}{2}\theta^a\,\mathrm{tr}\left[\Sigma^\dagger T^a(\partial^\mu\Sigma)\right] \equiv \theta^a j_L^{a\mu}
\end{aligned}$$

Thus, the eight Noether's currents for $SU(3)_L$ are

$$j_L^{a\mu} = -i\frac{f^2}{2}\mathrm{tr}\left[\Sigma^\dagger T^a(\partial^\mu \Sigma)\right] \qquad (a = 1, \cdots, 8)$$

Proof for the relation $\partial^\mu \Sigma^\dagger = -\Sigma^\dagger(\partial^\mu \Sigma)\Sigma^\dagger$:

$$\Sigma^\dagger \Sigma = 1 \quad \Rightarrow \quad 0 = \partial^\mu(\Sigma^\dagger \Sigma) = (\partial^\mu \Sigma^\dagger)\Sigma + \Sigma^\dagger(\partial^\mu \Sigma)$$
$$\Rightarrow \quad \partial^\mu \Sigma^\dagger = -\Sigma^\dagger(\partial^\mu \Sigma)\Sigma^\dagger \qquad \surd$$

$$\Sigma = \exp\left[\frac{2i}{f}\pi(x)\right] = 1 + \frac{2i}{f}\pi(x) + \mathcal{O}\left(\frac{1}{f^2}\pi^2\right)$$

$$\Rightarrow \quad \partial^\mu \Sigma = \frac{2i}{f}\partial^\mu \pi + \mathcal{O}\left(\frac{1}{f^2}\pi^2\right)$$

$$\Sigma^\dagger = \exp\left[-\frac{2i}{f}\pi(x)\right] = 1 - \frac{2i}{f}\pi(x) + \mathcal{O}\left(\frac{1}{f^2}\pi^2\right)$$

$$\begin{aligned}
\Rightarrow \quad j_L^{a\mu} &= -i\frac{f^2}{2}\mathrm{tr}\left[\Sigma^\dagger T^a(\partial^\mu \Sigma)\right] \\
&= -i\frac{f^2}{2}\mathrm{tr}\left[\left(1 - \frac{2i}{f}\pi(x) + \mathcal{O}\left(\frac{1}{f^2}\pi^2\right)\right) T^a \left(\frac{2i}{f}\partial^\mu \pi + \mathcal{O}\left(\frac{1}{f^2}\pi^2\right)\right)\right] \\
&= -i\frac{f^2}{2} \cdot \left[\frac{2i}{f}\mathrm{tr}\left[T^a(\partial^\mu \pi)\right] + \mathcal{O}\left(\frac{1}{f^2}\pi^2\right)\right] \\
&= f\,\mathrm{tr}\left[T^a(\partial^\mu \pi)\right] + \mathcal{O}\left(\frac{1}{f^0}\pi^2\right)
\end{aligned}$$

That is, we get the Noether's currents for $SU(3)_L$ as

$$j_L^{a\mu} = f\,\mathrm{tr}\left[T^a(\partial^\mu \pi)\right] + \mathcal{O}\left(\frac{1}{f^0}\pi^2\right) \qquad (a = 1, \cdots, 8)$$

In particular,

$$j_L^{1\mu} + ij_L^{2\mu} = f\,\mathrm{tr}\left[(T^1 + iT^2)(\partial^\mu \pi)\right] + \mathcal{O}\left(\frac{1}{f^0}\pi^2\right)$$

Let us figure out the trace:

$$\begin{aligned}
&\mathrm{tr}\left[(T^1 + iT^2)(\partial^\mu \pi)\right] \\
&= \mathrm{tr}\left[\begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{pmatrix} \partial^\mu \frac{1}{2} \begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}\right] \\
&= \frac{1}{\sqrt{2}}\partial^\mu \pi^-
\end{aligned}$$

Thus,

$$j_L^{1\mu} + i j_L^{2\mu} = \frac{f}{\sqrt{2}} \partial^\mu \pi^- + \mathcal{O}\left(\frac{1}{f^0}\pi^2\right)$$

$$\Rightarrow \quad \bar{u}\gamma^\mu(1-\gamma_5)d = 2(j_L^{1\mu} + i j_L^{2\mu}) = \sqrt{2} f \partial^\mu \pi^- + \mathcal{O}\left(\frac{1}{f^0}\pi^2\right)$$

Comparing this equation with

$$\langle 0|\bar{u}\gamma^\mu(1-\gamma_5)d|\pi^-(p)\rangle \equiv i\sqrt{2} f_\pi p^\mu$$

we see that to this order,

$$f = f_\pi = 93 \text{ MeV}$$

For semileptonic decays, the weak operator can be factorized into a leptonic matrix element and a hadronic matrix element of an $SU(3)_L$ symmetry current.

In this case, we can match quark operators with operators in the chiral Lagrangian.

We can also match quark operators with pion operators by dimensional analysis:

$$\Sigma_{ij} \propto \langle q_i \bar{q}_j \rangle = \left\langle \begin{pmatrix} u \\ d \\ s \end{pmatrix} \begin{pmatrix} \bar{u} & \bar{d} & \bar{s} \end{pmatrix} \right\rangle = \begin{pmatrix} \langle \bar{u}u \rangle & \langle \bar{d}u \rangle & \langle \bar{s}u \rangle \\ \langle \bar{u}d \rangle & \langle \bar{d}d \rangle & \langle \bar{s}d \rangle \\ \langle \bar{u}s \rangle & \langle \bar{d}s \rangle & \langle \bar{s}s \rangle \end{pmatrix}$$

$$\Sigma = \exp\left[\frac{2i}{f}\pi(x)\right] = 1 + \frac{2i}{f}\pi(x) + \mathcal{O}\left(\frac{1}{f^2}\pi^2\right)$$

$$= 1 + \frac{2i}{f}\begin{pmatrix} * & * & * \\ \frac{\pi^-}{\sqrt{2}} & * & * \\ * & * & * \end{pmatrix} + \mathcal{O}\left(\frac{1}{f^2}\pi^2\right)$$

Comparing the above equations, we can match $\bar{u}(\cdots)d$ to π^-.

The other factors or derivatives will be fixed by Lorentz invariance and dimensional analysis.

Lorentz index: $\gamma^\mu \to \partial^\mu \quad \Rightarrow \quad \bar{u}\gamma^\mu(1-\gamma_5)d \propto \partial^\mu \pi^-$

On the other hand,

$$[u] = [d] = \frac{3}{2} \quad \Rightarrow \quad [\bar{u}\gamma^\mu(1-\gamma_5)d] = 3$$
$$[\partial^\mu] = [\pi^-] = 1 \quad \Rightarrow \quad \left[\partial^\mu \pi^-\right] = 2$$

Thus, by dimensional analysis we have

$$\bar{u}\gamma^\mu(1-\gamma_5)d \sim f\partial^\mu \pi^- \quad \text{with} \quad [f] = 1$$

Shift symmetry of pions

Up to now, we have only discussed operators in the chiral Lagrangian which are invariant.

These are derivative interaction operators.

Recall that without explicit chiral symmetry breaking, there would be a shift symmetry of the pion fields.

The $SU(3)_L$ symmetry $\Sigma \to e^{iT^a\theta^a}\Sigma$ corresponds to shift the fields as

$$\pi^a \to \pi^a + \frac{f}{2}\theta^a + \mathcal{O}\left((1/f)^0\right)$$

$$L = e^{iT^a\theta^a}, \quad \Sigma = e^{\frac{2i}{f}\pi(x)} = e^{\frac{2i}{f}T^a\pi^a(x)}$$

$$\Rightarrow \quad \Sigma \to L\Sigma = e^{iT^a\theta^a} e^{\frac{2i}{f}T^b\pi^b}$$

Using Baker-Compbell-Hausdorff formula:

$$e^X e^Y = e^Z$$
$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \cdots$$

with $X = iT^a\theta^a,\ Y = \frac{2i}{f}T^b\pi^b$, we obtain

$$Z = iT^a\theta^a + \frac{2i}{f}T^a\pi^a + \frac{1}{2}\left[iT^a\theta^a, \frac{2i}{f}T^b\pi^b\right] + \cdots$$
$$= \frac{2i}{f}T^a\left(\pi^a + \frac{f}{2}\theta^a\right) + \mathcal{O}\left(\frac{\theta^a}{f}\right) = \frac{2i}{f}T^a\left[\pi^a + \frac{f}{2}\theta^a + \mathcal{O}\left((1/f)^0\theta^a\right)\right]$$
$$\Rightarrow \quad \pi^a \to \pi^a + \frac{f}{2}\theta^a + \mathcal{O}\left((1/f)^0\theta^a\right)$$

That is, derivative interactions are a result of the shift symmetry.

In the literature, this is called a nonlinearly realized symmetry, *i.e.*, a spontaneously broken symmetry.

A theory of massless particles with nontrivial interactions at zero momentum transfer (such as QCD, in which gluons are massless) would suffer severe infrared divergences.

So if the interactions had not been purely derivative, the theory would either not make sense, or become nonperturbative like QCD.

However, when explicit chiral symmetry breaking is included, not all vacua are equivalent, the massless Goldstone bosons become massive "pseudo-Goldstone bosons" (PGBs), and acquire nonderivative interactions.

Explicit symmetry breaking: quark masses

To describe the meson masses, we need to include the quark mass matrix $M = \mathrm{diag}(m_u,\ m_d,\ m_s)$ in the chiral Lagrangian.

To be dimensionless, each power of M will be accompanied by $1/\Lambda$.

The leading operator involving quark masses is given by

$$\begin{aligned}\mathcal{L}_M &= \Lambda^2 f^2 \left[\frac{c}{2}\mathrm{tr}\left(\frac{1}{\Lambda}M\Sigma\right) + \mathrm{h.c.}\right] \\ &= \frac{f^2}{2}c\Lambda\,\mathrm{tr}(M\Sigma + \mathrm{h.c.}) = \frac{f^2}{2}c\Lambda\,\mathrm{tr}[M(\Sigma + \Sigma^\dagger)]\end{aligned}$$

where c is an unknown dimensionless coefficient. The red part is dimension 4, while the blue part is dimensionless.

Let us define $\tilde{\Lambda} \equiv c\Lambda = \mathcal{O}(\Lambda)$, then this term becomes

$$\mathcal{L}_M = \frac{f^2}{2}\tilde{\Lambda}\,\mathrm{tr}(M\Sigma + \mathrm{h.c.}) = \frac{f^2}{2}\tilde{\Lambda}\,\mathrm{tr}[M(\Sigma + \Sigma^\dagger)]$$

Expanding $\mathcal{L}_M$ to second order in $\pi(x)$, we get

$$\mathcal{L}_M \simeq -m_{\pi^+}^2\pi^+\pi^- - m_{K^+}^2 K^+K^- - m_{K^0}^2 K^0\bar{K}^0 - \frac{1}{2}\begin{pmatrix}\pi^0 & \eta\end{pmatrix} M_0^2 \begin{pmatrix}\pi^0 \\ \eta\end{pmatrix}$$

with

$$m_{\pi^+}^2 = \tilde{\Lambda}(m_u + m_d), \quad m_{K^+}^2 = \tilde{\Lambda}(m_u + m_s), \quad m_{K^0}^2 = \tilde{\Lambda}(m_d + m_s)$$

$$M_0^2 = \tilde{\Lambda}\begin{pmatrix} m_u + m_d & \frac{m_u - m_d}{\sqrt{3}} \\ \frac{m_u - m_d}{\sqrt{3}} & \frac{1}{3}(m_u + m_d + 4m_s)\end{pmatrix}$$

Derivation of the meson masses from $\mathcal{L}_M = \frac{f^2}{2}\tilde{\Lambda}\,\mathrm{tr}[M(\Sigma+\Sigma^\dagger)]$:

$$\mathcal{L}_M = \frac{f^2}{2}\tilde{\Lambda}\,\mathrm{tr}(M\Sigma + h.c.) \quad = \mathrm{tr}(M\Sigma + \Sigma^\dagger M^\dagger) = \mathrm{tr}[M(\Sigma+\Sigma^\dagger)]$$

$$= \frac{f^2}{2}\tilde{\Lambda}\,\mathrm{tr}[M(\Sigma+\Sigma^\dagger)]$$

$$M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}, \qquad \Sigma = e^{\frac{2i}{f}\pi(x)} = 1 + \frac{2i}{f}\pi - \frac{2}{f^2}\pi^2 + \cdots$$

$$\Sigma^\dagger = e^{-\frac{2i}{f}\pi(x)} = 1 - \frac{2i}{f}\pi - \frac{2}{f^2}\pi^2 + \cdots$$

$$\Rightarrow \Sigma + \Sigma^\dagger \supset -\frac{4}{f^2}\pi^2 \qquad \Rightarrow \mathrm{tr}[M(\Sigma+\Sigma^\dagger)] = -\frac{4}{f^2}\mathrm{tr}(M\pi^2)$$

$$\pi(x) = \frac{1}{2}\begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix} \Rightarrow \begin{aligned} (\pi^2)_{11} &= \frac{1}{4}\left[\left(\pi^0+\frac{\eta}{\sqrt{3}}\right)^2 + 2\pi^+\pi^- + 2K^+K^-\right] \\ (\pi^2)_{22} &= \frac{1}{4}\left[2\pi^+\pi^- + \left(-\pi^0+\frac{\eta}{\sqrt{3}}\right)^2 + 2K^0\bar{K}^0\right] \\ (\pi^2)_{33} &= \frac{1}{4}\left[2K^+K^- + 2K^0\bar{K}^0 + \frac{4}{3}\eta^2\right] \end{aligned}$$

$$\Rightarrow \mathcal{L}_M = \frac{f^2}{2}\lambda \underbrace{\mathrm{tr}[M(\Sigma+\Sigma^\dagger)]}_{\substack{\| \\ -\frac{4}{f^2}\mathrm{tr}(M\pi^2)}} = -\frac{\tilde{\Lambda}}{2}\, 4\, \underbrace{\mathrm{tr}(M\pi^2)}_{\substack{\| \\ m_u(\pi^2)_{11}+m_d(\pi^2)_{22}+m_s(\pi^2)_{33}}}$$

$$= -\frac{\tilde{\Lambda}}{2}\Big\{ m_u\Big[\Big(\pi^0+\frac{\eta}{\sqrt{3}}\Big)^2 + 2\pi^+\pi^- + 2K^+K^-\Big]$$

$$+ m_d\Big[2\pi^+\pi^- + \Big(-\pi^0+\frac{\eta}{\sqrt{3}}\Big)^2 + 2K^0\bar{K}^0\Big]$$

$$+ m_s\Big(2K^+K^- + 2K^0\bar{K}^0 + \frac{4}{3}\eta^2\Big)\Big\}$$

$$\supset -\underbrace{\lambda(m_u+m_d)}_{\substack{\| \\ m_\pi^2}}\pi^+\pi^- - \underbrace{\lambda(m_u+m_s)}_{\substack{\| \\ m_{K^+}^2}}K^+K^- - \underbrace{\tilde{\Lambda}(m_d+m_s)}_{\substack{\| \\ m_{K^0}^2}}K^0\bar{K}^0 \qquad \checkmark$$

$$\mathcal{L}_M \supset -\frac{\tilde{\Lambda}}{2} m_u \underbrace{\left(\pi^0 + \frac{\eta}{\sqrt{3}}\right)^2}_{= (\pi^0)^2 + \frac{\eta^2}{3} + \frac{2}{\sqrt{3}}\pi^0\eta} - \frac{\tilde{\Lambda}}{2} m_d \underbrace{\left(-\pi^0 + \frac{\eta}{\sqrt{3}}\right)^2}_{= (\pi^0)^2 + \frac{\eta^2}{3} - \frac{2}{\sqrt{3}}\pi^0\eta} - \frac{\tilde{\Lambda}}{2} m_s \frac{4}{3}\eta^2$$

$\Rightarrow$ Coefficient of $(\pi^0)^2$: $-\frac{\tilde{\Lambda}}{2}(m_u + m_d)$

η^2: $-\frac{\tilde{\Lambda}}{2}\left(\frac{m_u}{3} + \frac{m_d}{3} + \frac{4m_s}{3}\right)$

$\pi^0\eta$: $-\frac{\tilde{\Lambda}}{2}\left(\frac{2}{\sqrt{3}}m_u - \frac{2}{\sqrt{3}}m_d\right) = -\tilde{\Lambda}\frac{(m_u - m_d)}{\sqrt{3}}$

$$\Rightarrow \mathcal{L}_M \supset -\frac{1}{2}(\pi^0, \eta) M_0^2 \begin{pmatrix} \pi^0 \\ \eta \end{pmatrix} \quad \text{with} \quad M_0^2 = \tilde{\Lambda} \begin{pmatrix} m_u + m_d & \frac{(m_u - m_d)}{\sqrt{3}} \\ \frac{(m_u - m_d)}{\sqrt{3}} & \frac{m_u + m_d + 4m_s}{3} \end{pmatrix} \checkmark$$

Note that (i) the squares of the meson masses are proportional to quark masses;

(ii) π^0-η mixing is isospin breaking and proportional to $(m_u - m_d)$;

(iii) expanding in powers of $(m_u - m_d)$, m_η^2 and $m_{\pi^0}^2$ are given by the diagonal entries of M_0^2, up to corrections of $\mathcal{O}\left((m_u - m_d)^2\right)$;

(iv) we cannot directly relate quark and meson masses because of the unknown coefficient $\tilde{\Lambda}$.

Ignoring isospin breaking due to electromagnetism and the difference $m_u \neq m_d$, the meson masses obey the Gell-Mann Okuba formula:

$$3m_\eta^2 + m_\pi^2 = 4m_K^2$$

Derivation of the Gell-Mann Okuba formula:

$$m_{\pi^+}^2 = \tilde{\Lambda}(m_u + m_d), \quad m_{K^+}^2 = \tilde{\Lambda}(m_u + m_s), \quad m_{K^0}^2 = \tilde{\Lambda}(m_d + m_s),$$

$$m_{\pi^0}^2 = \tilde{\Lambda}(m_u + m_d) + \mathcal{O}((m_d - m_u)^2),$$

$$m_\eta^2 = \tilde{\Lambda}\,\frac{m_u + m_d + 4m_s}{3} + \mathcal{O}((m_d - m_u)^2)$$

If $m_u = m_d$, then

$$m_{\pi^+}^2 = m_{\pi^0}^2 = \tilde{\Lambda}\cdot 2m_u,$$

$$m_{K^+}^2 = m_{K^0}^2 = \tilde{\Lambda}(m_u + m_s),$$

$$m_\eta^2 = \tilde{\Lambda}\,\frac{2m_u + 4m_s}{3}$$

$$\begin{aligned} 3m_\eta^2 + m_\pi^2 &= \tilde{\Lambda}\left[(2m_u + 4m_s) + 2m_u\right] \\ &= 4\tilde{\Lambda}(m_u + m_s) \\ &= 4m_K^2 \quad \checkmark \end{aligned}$$

Explicit symmetry breaking: electromagnetism

To include electromagnetism into the chiral Lagrangian, we first go back to QCD and ask what currents out of $j_L^{a\mu}$ and $j_R^{a\mu}$ couple to the photon.

This is easy: the electromagnetic current coupling to the photon is given by

$$j^\mu_{em} = e\bar{q}_L\gamma^\mu Q_L q_L + e\bar{q}_R\gamma^\mu Q_R q_R$$

$$\text{with} \quad Q_L = Q_R = \mathrm{diag}\left(\frac{2}{3},\ -\frac{1}{3},\ -\frac{1}{3}\right)$$

$$j^\mu_{em} = \bar{q}\gamma^\mu e Q q \quad \text{with} \quad Q = \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}$$

$$= e\bar{q}\gamma^\mu Q (P_L + P_R) q$$

$$= e\bar{q}_L\gamma^\mu Q_L q_L + e\bar{q}_R\gamma^\mu Q_R q_R, \quad Q_L = Q_R = Q$$

★ For any field $\phi(x)$ carrying charge q under some local $U(1)$ symmetry, its $U(1)$ transformation and covariant derivative are

$$\phi(x) \longrightarrow e^{iq\alpha(x)}\,\phi(x)$$

$$D_\mu \phi = (\partial_\mu - iqA_\mu)\,\phi$$

The local $U(1)$ symmetry requares

$$\partial_\mu - iqA_\mu \longrightarrow \underbrace{e^{iq\alpha(x)}\,(\partial_\mu - iqA_\mu)\,e^{-iq\alpha(x)}}_{=\ \partial_\mu + \underbrace{e^{iq\alpha(x)}(\partial_\mu e^{-iq\alpha(x)})}_{= -iq\partial_\mu\alpha} - iqA_\mu}$$

$$\Rightarrow \ \underline{A_\mu \to A_\mu + \partial_\mu \alpha}$$

★ If $\Sigma \to e^{ieQ_L\alpha(x)}\Sigma$ with $e^{ieQ_L\alpha(x)} \in U(1)_{em} \subset SU(3)_L$,

then
$$D_\mu\Sigma = (\partial_\mu - ieQ_L A_\mu)\Sigma$$

★ If $\Sigma \longrightarrow \Sigma\, e^{-ieQ_R\alpha(x)}$ with $e^{ieQ_R\alpha(x)} \in U(1)_{em} \subset SU(3)_R$,

then
$$\begin{aligned} D_\mu\Sigma &= \partial_\mu\Sigma + ieA_\mu\Sigma Q_R \\ &= \Sigma(\overleftarrow{\partial}_\mu + ieQ_R A_\mu) \end{aligned}$$

★ If $\Sigma \to e^{ieQ_L\alpha(x)}\Sigma\, e^{-ieQ_R\alpha(x)}$ with $U(1)_{em} \subset SU(3)_V$,

then
$$D_\mu\Sigma = \partial_\mu\Sigma - ieA_\mu(Q_L\Sigma - \Sigma Q_R)$$

The $U(1)_{em}$ gauge symmetry determines the covariant derivative of Σ to be

$$D_\mu \Sigma = \partial_\mu \Sigma - ieA_\mu (Q_L \Sigma - \Sigma Q_R)$$

since $\Sigma \to L\Sigma R^\dagger$ under $SU(3)_L \times SU(3)_R$, and $U(1)_{em} \subset SU(3)_V \subset SU(3)_L \times SU(3)_R$.

When we set the Σ field to its VEV, $\Sigma = 1$, the photon term drops out of the covariant derivative, which is to say that the vacuum does not break the $U(1)_{em}$ symmetry.

$Q_{L,R}$ in general do not commute with L and R, so terms involving $D_\mu \Sigma$, for example $\mathrm{tr}[(D_\mu \Sigma^\dagger)(D^\mu \Sigma)]$, explicitly break the $SU(3)_L \times SU(3)_R$ chiral symmetry.

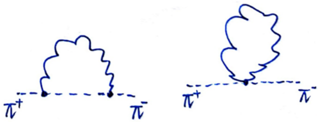
However, we can promote $Q_{L,R}$ to be spurions: we define their chiral transformations as

$$Q_L \to L Q_L L^\dagger, \quad Q_R \to R Q_R R^\dagger$$

such that

$$D_\mu \Sigma \to L(D_\mu \Sigma) R^\dagger$$

If we want to compute the electromagnetic contribution to the π^+–π^0 mass splitting to order α, we naturally look at the two 1-loop diagrams we encounter in scalar QED.



These two 1-loop diagrams are apparently quadratically divergent, which means they need a counterterm which would contribute to the pion mass squared.

- This counterterm should be hermitian, so it will contain the trace of Σ and $\Sigma^\dagger$.
- This counterterm should be proportional to e^2, and thus it will be proportional to $(eQ_L)^2$, $(eQ_R)^2$ or $eQ_L eQ_R$.
- This counterterm should be chirally symmetric when we regard $Q_{L,R}$ as spurions:

$$Q_L \to L Q_L L^\dagger, \quad Q_R \to R Q_R R^\dagger$$

The only possible counterterm satisfying these three requirements is of the form

$$\mathcal{L}_\alpha = \xi f^4 e^2 \mathrm{tr}\left(Q_L \Sigma Q_R \Sigma^\dagger\right)$$

where the coefficient $\xi = \mathcal{O}(1)$ needs to be fit to data or computed using lattice QCD.

★ hermitian $\Rightarrow$ $\mathcal{L}_\alpha \propto \mathrm{tr}\left[(\cdots)\Sigma(\cdots)\Sigma^\dagger\right]$

★ $\mathcal{L}_\alpha \propto e^2$

★ chiral symmetry: $\begin{cases} \Sigma \to L\Sigma R^\dagger \\ \Sigma^\dagger \to R\Sigma^\dagger L^\dagger \\ Q_L \to L Q_L L^\dagger, \quad Q_R \to R Q_R R^\dagger \end{cases}$ $\Rightarrow$ $\mathcal{L}_\alpha \propto \mathrm{tr}\left(\overset{Q_L}{\downarrow}\Sigma \overset{Q_R}{\downarrow}\Sigma^\dagger\right)$

✓

If we use the $\overline{\text{MS}}$ scheme in Landau gauge, then the 1-loop diagrams vanish in the low energy limit, and we are left only with the direct contribution from the counterterm.

Expanding it to second order in meson fields we get

$$\begin{aligned}\mathcal{L}_\alpha &= -\xi f^4 e^2 \frac{2}{f^2} \mathrm{tr}\left(Q_L[\pi,[\pi,Q_R]]\right) \\ &= -2\xi e^2 f^2 (\pi^+\pi^- + K^+K^-)\end{aligned}$$

This simple result says that the meson mass^2 gets shifted by a constant amount proportional to its charged squared.

$$\mathcal{L}_\alpha = \xi f^4 e^2 \,\mathrm{tr}(Q_L \underbrace{\Sigma Q_R \Sigma^\dagger}_{a}), \quad \Sigma = e^{\frac{2i}{f}\pi(x)} \Rightarrow \Sigma^\dagger = e^{-\frac{2i}{f}\pi(x)}$$

Using Baker-Campbell-Hausdorff formula:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A,[A,B]] + \cdots$$

with $A = \frac{2i}{f}\pi(x)$, $B = Q_R$, we obtain

$$\Sigma Q_R \Sigma^\dagger = e^{\frac{2i}{f}\pi(x)} Q_R \, e^{-\frac{2i}{f}\pi(x)}$$

$$= \underbrace{Q_R}_{a1} + \underbrace{\frac{2i}{f}[\pi(x), Q_R]}_{a2} + \underbrace{\frac{1}{2}\left[\frac{2i}{f}\pi, \left[\frac{2i}{f}\pi, Q_R\right]\right]}_{a3} + \cdots$$

$\mathrm{tr}(Q_L \, (a1)) = \mathrm{tr}(Q_L Q_R)$ is just a senseless constant.

$$\mathrm{tr}(Q_L \, (a2)) = \mathrm{tr}\left(Q_L \frac{2i}{f}[\pi(x), Q_R]\right) = \frac{2i}{f}\mathrm{tr}[Q_L(\pi Q_R - Q_R \pi)]$$

$$= \frac{2i}{f}\mathrm{tr}(\pi Q_R Q_L - \pi Q_L Q_R) = 0 \quad (\because \; Q_L = Q_R = Q)$$

$$\mathrm{tr}(Q_L \, (a3)) = \mathrm{tr}\left(Q_L \frac{1}{2}\left[\frac{2i}{f}\pi, \left[\frac{2i}{f}\pi, Q_R\right]\right]\right) = -\frac{2}{f^2}\mathrm{tr}(Q_L[\pi,[\pi,Q_R]])$$

$$\Rightarrow \mathcal{L}_\alpha = \xi f^4 e^2 \underbrace{\mathrm{tr}(Q_L \Sigma Q_R \Sigma^\dagger)}_{\shortparallel}$$

$$\left(-\frac{2}{f^2}\right)\mathrm{tr}(Q_L[\pi,[\pi,Q_R]]) + \mathcal{O}\left(\frac{1}{f^3}\right)$$

$$= -2\xi e^2 f^2 \, \mathrm{tr}(Q_L[\pi,[\pi,Q_R]]) + \mathcal{O}(f)$$

✓

Now let us show the following relations:

$$\mathrm{tr}(Q_L[\pi, [\pi, Q_R]]) = 2\,\mathrm{tr}(Q^2\pi^2 - Q\pi Q\pi) \qquad ①$$

$$\mathrm{tr}(Q^2\pi^2 - Q\pi Q\pi) = \frac{1}{2}(\pi^+\pi^- + K^+K^-) \qquad ②$$

Proof for ①:

$$Q_L = Q_R = Q \quad \Rightarrow \quad \mathrm{tr}(Q_L[\pi, [\pi, Q_R]]) = \mathrm{tr}(Q[\pi, [\pi, Q]])$$

$$= \mathrm{tr}[Q(\pi[\pi, Q] - [\pi, Q]\pi)]$$

$$= \mathrm{tr}[Q\pi(\pi Q - Q\pi) - Q(\pi Q - Q\pi)\pi]$$

$$= \mathrm{tr}(Q^2\pi^2 - Q\pi Q\pi - Q\pi Q\pi + Q^2\pi^2)$$

$$= 2\,\mathrm{tr}(Q^2\pi^2 - Q\pi Q\pi) \quad \checkmark$$

Proof for 2:

$$Q=\begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix} \Rightarrow Q^2=\begin{pmatrix} \frac{4}{9} & & \\ & \frac{1}{9} & \\ & & \frac{1}{9} \end{pmatrix}$$

$$\pi=\frac{1}{2}\begin{pmatrix} \pi^0+\frac{\eta}{\sqrt{3}} & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0+\frac{\eta}{\sqrt{3}} & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

$$\Rightarrow \begin{cases} (\pi^2)_{11}=\frac{1}{4}\left[\left(\pi^0+\frac{\eta}{\sqrt{3}}\right)^2+2\pi^+\pi^-+2K^+K^-\right] \\ (\pi^2)_{22}=\frac{1}{4}\left[2\pi^+\pi^-+\left(-\pi^0+\frac{\eta}{\sqrt{3}}\right)^2+2K^0\bar{K}^0\right] \\ (\pi^2)_{33}=\frac{1}{4}\left(2K^+K^-+2K^0\bar{K}^0+\frac{4}{3}\eta^2\right) \end{cases}$$

$$
\begin{aligned}
\Rightarrow \operatorname{tr}(Q^2\pi^2) &= \frac{4}{9}(\pi^2)_{11} + \frac{1}{9}(\pi^2)_{22} + \frac{1}{9}(\pi^2)_{33} \\
&= \frac{4}{9}\,\frac{1}{4}\left[\left(\pi^0+\frac{\eta}{\sqrt{3}}\right)^2 + 2\pi^+\pi^- + 2K^+K^-\right] \\
&\quad + \frac{1}{9}\,\frac{1}{4}\left[2\pi^+\pi^- + \left(-\pi^0+\frac{\eta}{\sqrt{3}}\right)^2 + 2K^0\bar{K}^0\right] \\
&\quad + \frac{1}{9}\,\frac{1}{4}\left(2K^+K^- + 2K^0\bar{K}^0 + \frac{4}{3}\eta^2\right) \\
&= \underbrace{\frac{1}{9}\left(\pi^0+\frac{\eta}{\sqrt{3}}\right)^2}_{\text{[Cancel-1]}} + \underbrace{\frac{1}{36}\left(-\pi^0+\frac{\eta}{\sqrt{3}}\right)^2}_{\text{[Cancel-2]}} + \underbrace{\frac{1}{36}\cdot\frac{4}{3}\eta^2}_{\text{[Cancel-3]}} \\
&\quad + \left(\frac{2}{9}+\frac{1}{18}\right)\pi^+\pi^- + \left(\frac{2}{9}+\frac{1}{18}\right)K^+K^- + \left(\frac{1}{18}+\frac{1}{18}\right)K^0\bar{K}^0
\end{aligned}
$$

On the other hand, we get

$$Q\pi = \frac{1}{2}\begin{pmatrix} \frac{2}{3}\left(\pi^0+\frac{\eta}{\sqrt{3}}\right) & \frac{2}{3}\sqrt{2}\,\pi^+ & \frac{2}{3}\sqrt{2}\,K^+ \\ -\frac{1}{3}\sqrt{2}\,\pi^- & -\frac{1}{3}\left(-\pi^0+\frac{\eta}{\sqrt{3}}\right) & -\frac{1}{3}\sqrt{2}\,K^0 \\ -\frac{1}{3}\sqrt{2}\,K^- & -\frac{1}{3}\sqrt{2}\,\bar{K}^0 & \frac{1}{3}\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

$$\Rightarrow \operatorname{tr}(Q\pi Q\pi) = \frac{1}{4}\Big[\underbrace{\frac{4}{9}\left(\pi^0+\frac{\eta}{\sqrt{3}}\right)^2}_{\text{[Cancel-1]}} - \frac{4}{9}\pi^+\pi^- - \frac{4}{9}K^+K^-$$

$$-\frac{4}{9}\pi^+\pi^- + \underbrace{\frac{1}{9}\left(-\pi^0+\frac{\eta}{\sqrt{3}}\right)^2}_{\text{[Cancel-2]}} + \frac{2}{9}K^0\bar{K}^0$$

$$-\frac{4}{9}K^+K^- + \frac{2}{9}K^0\bar{K}^0 + \underbrace{\frac{1}{9}\frac{4}{3}\eta^2}_{\text{[Cancel-3]}}\Big]$$

Now let us consider $\mathrm{tr}(Q^2\pi^2) - \mathrm{tr}(Q\pi Q\pi)$.

Coefficient of $\pi^+\pi^-$: $\left(\frac{2}{9}+\frac{1}{18}\right)+\frac{1}{4}\cdot\frac{4}{9}\cdot 2 = \frac{4}{9}+\frac{1}{18} = \frac{9}{18} = \frac{1}{2}$ ✓

K^+K^-: $\left(\frac{2}{9}+\frac{1}{18}\right)+\frac{1}{4}\cdot\frac{4}{9}\cdot 2 = \frac{1}{2}$ ✓

$K^0\bar{K}^0$: $\left(\frac{1}{18}+\frac{1}{18}\right)-\frac{1}{4}\cdot\frac{2}{9}\cdot 2 = \frac{1}{9}-\frac{1}{9} = 0$ ✓

$$\Rightarrow \quad \mathrm{tr}(Q^2\pi^2) - \mathrm{tr}(Q\pi Q\pi) = \frac{1}{2}(\pi^+\pi^- + K^+K^-) \quad ✓$$

Therefore, $\mathcal{L}_\alpha \simeq -2\xi e^2 f^2 \underbrace{\mathrm{tr}(Q_L[\pi,[\pi,Q_R]])}_{= 2\mathrm{tr}(Q^2\pi^2 - Q\pi Q\pi) = (\pi^+\pi^- + K^+K^-)}$

$$\Rightarrow \quad \mathcal{L}_\alpha \simeq -2\xi e^2 f^2 (\pi^+\pi^- + K^+K^-) \quad ✓$$

Proof for

$$\boxed{\mathrm{tr}(Q^2\pi^2) - \mathrm{tr}(Q\pi Q\pi) = \frac{1}{2}(\pi^+\pi^- + K^+K^-)}$$

by *Mathematica*:

In[1]:= $\mathbf{pifield} = \frac{1}{2}\begin{pmatrix} \mathtt{pi0} + \frac{\mathtt{eta}}{\sqrt{3}} & \sqrt{2}\ \mathtt{piplus} & \sqrt{2}\ \mathtt{Kplus} \\ \sqrt{2}\ \mathtt{piminus} & -\mathtt{pi0} + \frac{\mathtt{eta}}{\sqrt{3}} & \sqrt{2}\ \mathtt{Kzero} \\ \sqrt{2}\ \mathtt{Kminus} & \sqrt{2}\ \mathtt{Kzerobar} & -\frac{2\,\mathtt{eta}}{\sqrt{3}} \end{pmatrix}$; **Tr[pifield.pifield] // Simplify**

Out[1]= $\frac{1}{2}\left(\mathrm{eta}^2 + 2\,\mathrm{Kminus\ Kplus} + 2\,\mathrm{Kzero\ Kzerobar} + \mathrm{pi0}^2 + 2\,\mathrm{piminus\ piplus}\right)$

In[2]:= $\mathbf{Q = DiagonalMatrix}\left[\left\{\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right\}\right]$;

Tr[Q.Q.pifield.pifield] - Tr[Q.pifield.Q.pifield] // Simplify

Out[2]= $\frac{1}{2}\left(\mathrm{Kminus\ Kplus} + \mathrm{piminus\ piplus}\right)$

Thus, to leading order in α and the quark masses, the meson masses are

$$m_{\pi^+}^2 = \tilde{\Lambda}(m_u + m_d) + \frac{\alpha}{4\pi}\Delta^2$$
$$m_{K^+}^2 = \tilde{\Lambda}(m_u + m_s) + \frac{\alpha}{4\pi}\Delta^2$$

where $\Delta^2 \equiv 2\xi(4\pi f)^2$ is a parameter whose value we cannot predict.

$$\mathcal{L}_M = \frac{f^2}{2}\tilde{\lambda}\,\mathrm{tr}[M(\Sigma+\Sigma^\dagger)]$$

$$\supset -\tilde{\lambda}(m_u+m_d)\pi^+\pi^- - \tilde{\lambda}(m_u+m_s)K^+K^-$$

$$\mathcal{L}_\alpha \simeq -2\xi e^2 f^2(\pi^+\pi^- + K^+K^-)$$

$$\Rightarrow \quad \mathcal{L} \supset -\underbrace{\left[\tilde{\lambda}(m_u+m_d)+2\xi e^2 f^2\right]}_{= m_{\pi^+}^2}\pi^+\pi^- - \underbrace{\left[\tilde{\lambda}(m_u+m_s)+2\xi e^2 f^2\right]}_{= m_{K^+}^2}K^+K^-$$

$$\frac{\overset{\frac{e^2}{4\pi}}{\overset{\parallel}{\alpha}}}{4\pi}\,\overset{2\xi(4\pi f)^2}{\overset{\parallel}{\Delta^2}} = \frac{e^2}{\cancel{(4\pi)^2}}\,2\xi(\cancel{4\pi}f)^2 = 2\xi e^2 f^2 \quad \checkmark$$

We can manipulate the formulas to make predictions for ratios of combinations of meson masses so that the unknown parameters Δ and $\tilde{\Lambda}$ drop out:

$$\frac{(m_{K^+}^2 - m_{K^0}^2) - (m_{\pi^+}^2 - m_{\pi^0}^2)}{m_{\pi^0}^2} = \frac{m_u - m_d}{m_u + m_d}$$

$$\frac{3m_\eta^2 - m_{\pi^0}^2}{m_{\pi^0}^2} = \frac{4m_s}{m_u + m_d}$$

$$m_{\pi^0}^2 \simeq \tilde{\Lambda}(m_u + m_d), \qquad m_\eta^2 \simeq \tilde{\Lambda}\,\frac{m_u + m_d + 4m_s}{3}$$

$$\Rightarrow \quad \frac{3m_\eta^2 - m_{\pi^0}^2}{m_{\pi^0}^2} \simeq \frac{4m_s}{m_u + m_d} \qquad \checkmark$$

$$m_{K^+}^2 = \tilde{\Lambda}(m_u + m_s) + \frac{\alpha}{4\pi}\Delta^2, \qquad m_{K^0}^2 = \tilde{\Lambda}(m_d + m_s)$$

$$\Rightarrow \quad m_{K^+}^2 - m_{K^0}^2 = \tilde{\Lambda}(m_u - m_d) + \frac{\alpha}{4\pi}\Delta^2$$

Together with $m_{\pi^+}^2 - m_{\pi^0}^2 = \frac{\alpha}{4\pi}\Delta^2$, we get

$$\frac{(m_{K^+}^2 - m_{K^0}^2) - (m_{\pi^+}^2 - m_{\pi^0}^2)}{m_{\pi^0}^2} = \frac{\tilde{\Lambda}(m_u - m_d)}{\tilde{\Lambda}(m_u + m_d)} = \frac{m_u - m_d}{m_u + m_d} \qquad \checkmark$$

Plugging in the measured meson masses, these formulas can determine the ratios of quark masses in QCD:

$$\frac{m_u}{m_d} \simeq \frac{1}{2}, \quad \frac{m_d}{m_s} \simeq \frac{1}{20}$$

Lattice QCD calculation in the $\overline{\text{MS}}$ scheme gives

$$m_s \sim 100 \text{ MeV}$$

which implies

$$m_d \sim 5 \text{ MeV}, \quad m_u \sim 2.5 \text{ MeV}$$

Using *Mathematica* to **Solve** the equations:

$$\frac{(m_{K^+}^2 - m_{K^0}^2) - (m_{\pi^+}^2 - m_{\pi^0}^2)}{m_{\pi^0}^2} = \frac{m_u - m_d}{m_u + m_d} \quad \text{and} \quad \frac{3m_\eta^2 - m_{\pi^0}^2}{m_{\pi^0}^2} = \frac{4m_s}{m_u + m_d}$$

```
In[1]:= (*Define the meson masses in unit of MeV from the webpage: https://
         pdg.lbl.gov/2021/tables/rpp2021-sum-mesons.pdf *)
        MassPiPlus = 139.57039; MassPi0 = 134.9768; MassEta = 547.862;
        MassKPlus = 493.677;
        MassK0 = 497.611;

In[2]:= (* Solve the equation for the ratio x=mu/md *)
        solx = Solve[(MassKPlus^2 - MassK0^2 - (MassPiPlus^2 - MassPi0^2))/MassPi0^2 == (x - 1)/(x + 1), x]

Out[2]= {{x -> 0.558514}}

In[3]:= x = x /. solx[[1]]

Out[3]= 0.558514

In[4]:= (* Solve the equation for the ratio y=md/ms *)
        Solve[(3 MassEta^2 - MassPi0^2)/MassPi0^2 == 4/(x * y + y), y]

Out[4]= {{y -> 0.0530006}}
```

Power counting

The chiral Lagrangian is an EFT of QCD.

It consists of all local operators consistent with the symmetries of QCD, and there exists a power counting scheme that allows one to work to a given order, and to be able to make a reliable estimate of the errors arising from neglecting the subsequent orders.

Beyond the leading term $\mathcal{L}_0 = \frac{f^2}{4} \mathrm{tr}\left[(\partial\Sigma^\dagger)(\partial\Sigma)\right]$, one can write down an infinite number of chirally invariant operators which are higher powers in derivatives, as well as operators with more insertions of the quark mass matrix M.

The derivative expansion is in power of ∂/Λ, where $\Lambda \sim \mathcal{O}(1\ \mathrm{GeV})$ is the "chiral symmetry breaking scale".

Since the meson octet masses scale as

$$m_\pi^2 \sim \tilde{\Lambda} M$$

with quark mass M, and since for on-shell pions $p^2 \sim m_\pi^2$, it follows that one insertion of quark mass matrix is equivalent to two derivatives in the power counting.

That is, the chiral Lagrangian is a function of ∂/Λ and $\tilde{\Lambda} M/\Lambda^2$.

This power counting is consistent with the leading operator which can be written as

$$\mathcal{L}_0 = \underbrace{\Lambda^2 f^2}_{\text{dimension 4}} \underbrace{\left\{ \frac{1}{4\Lambda^2} \operatorname{tr}[(\partial\Sigma^\dagger)(\partial\Sigma)] \right\}}_{\text{dimensionless}}$$

That is, the chiral Lagrangian has a prefactor of $\Lambda^2 f^2$ and derivatives enter as ∂/Λ even in the leading operator.

Including electromagnetism is simply achieved by replacing the derivative $\partial\Sigma$ by the covariant derivative

$$D_\mu \Sigma = \partial_\mu \Sigma - ieA_\mu [Q, \Sigma]$$

Thus, the covariant derivative should come with a $1/Lambda$ factor, and the photon field enters as eA_μ/Λ.

Operators arising from electromagnetic loops involve two insertions of the quark charge matrix Q in a proper way, along with a loop factor $\alpha/(4\pi)$.

Therefore, the chiral Lagrangian takes the form

$$\mathcal{L} = \Lambda^2 f^2 \underbrace{\hat{\mathcal{L}}\left[\Sigma,\ D_\mu/\Lambda,\ \tilde{\lambda} M/\Lambda^2,\ \frac{\alpha}{4\pi} Q^2\right]}_{\text{dimensionless}}$$

where $\hat{\mathcal{L}}$ is a dimensionless sum of all local, chirally invariant operators (treating M and Q as spurions).

The coefficient of each term (except $\mathcal{L}_0$) is preceded by a dimensionless coefficient to be fit to experiments which we expect to be $\mathcal{O}(1)$.

This assumption allows one to estimate the size of higher order corrections.

For example, we assume $\mathcal{L} \supset \underbrace{\Lambda^2 f^2 \frac{\mathcal{O}(1)}{\Lambda^4}}_{\equiv L_1} \left(\mathrm{tr}[(\partial_\mu \Sigma^\dagger)(\partial^\mu \Sigma)]\right)^2$,

i.e. $\mathcal{L} \supset L_1 \left(\mathrm{tr}[(\partial_\mu \Sigma^\dagger)(\partial^\mu \Sigma)]\right)^2$ with $L_1 \sim \frac{\Lambda^2 f^2}{\Lambda^4} \mathcal{O}(1) = \frac{f^2}{\Lambda^2} \mathcal{O}(1) \sim 0.01$ $\quad (4\pi f)^2 \sim \Lambda^2$

Subleading order: $\mathcal{O}(p^4)$ chiral Lagrangian

It is straightforward to write down the subleading operators of the chiral Lagrangian.

The operators of $\mathcal{O}(p^4)$, $\mathcal{O}(p^2M)$ and $\mathcal{O}(M^2)$ are given by

$$
\begin{aligned}
\mathcal{L}_{p^4} = {} & L_1\left(\mathrm{tr}[(\partial_\mu\Sigma^\dagger)(\partial^\mu\Sigma)]\right)^2 + L_2\,\mathrm{tr}[(\partial_\mu\Sigma^\dagger)(\partial_\nu\Sigma)]\,\mathrm{tr}[(\partial^\mu\Sigma^\dagger)(\partial^\nu\Sigma)] \\
& + L_3\,\mathrm{tr}[(\partial_\mu\Sigma^\dagger)(\partial^\mu\Sigma)(\partial_\nu\Sigma^\dagger)(\partial^\nu\Sigma)] + L_4\,\mathrm{tr}[(\partial_\mu\Sigma^\dagger)(\partial^\mu\Sigma)]\,\mathrm{tr}(\chi\Sigma + \mathrm{h.c.}) \\
& + L_5\,\mathrm{tr}[(\partial_\mu\Sigma^\dagger)(\partial^\mu\Sigma)(\chi\Sigma + \mathrm{h.c.})] + L_6\left[\mathrm{tr}(\chi\Sigma + \mathrm{h.c.})\right]^2 \\
& + L_7\left[\mathrm{tr}(\chi\Sigma - \mathrm{h.c.})\right]^2 + L_8\,\mathrm{tr}(\chi\Sigma\chi\Sigma + \mathrm{h.c.})
\end{aligned}
$$

where $\chi \equiv 2\tilde{\Lambda}M$ with $\tilde{\Lambda}$ entered in $\mathcal{L}_M = \frac{f^2}{2}\tilde{\Lambda}\,\mathrm{tr}(M\Sigma + \mathrm{h.c.})$.

Additional operators involving $F_{\mu\nu}$ need to be considered when including electromagnetism.

Loop effects and naive dimension analysis

How big are radiative corrections in the chiral Lagrangian?

Let us rescale the terms in the action

$$S = \int d^4x \, \Lambda^2 f^2 \underbrace{\hat{\mathcal{L}}\left[\Sigma,\ D_\mu/\Lambda,\ \tilde{\Lambda} M/\Lambda^2,\ \frac{\alpha}{4\pi} Q^2\right]}_{\text{dimensionless}}$$

such that

$$\hat{x} = \Lambda x\ ,$$

$$\hat{M} = \frac{M\tilde{\Lambda}}{\Lambda^2}\ ,$$

$$\hat{A}_\mu = \frac{A_\mu}{\Lambda}$$

then the action becomes $S = \frac{f^2}{\Lambda^2} \int d^4\hat{x} \, \underbrace{\hat{\mathcal{L}}\left[\Sigma,\ \hat{D}_\mu,\ \hat{M},\ \frac{\alpha}{4\pi} Q^2\right]}_{\text{dimensionless}}$.

From

$$\exp\left(\frac{i}{\hbar}S\right) = \exp\left(\frac{i}{\hbar}\frac{f^2}{\Lambda^2}\int d^4\hat{x}\hat{\mathcal{L}}\right) = \exp\left[\frac{i}{(\hbar\Lambda^2/f^2)}\int d^4\hat{x}\hat{\mathcal{L}}\right]$$

we see that $\hbar$ is multiplied by Λ^2/f^2 in this theory.

Since we get a power of $\hbar$ with each loop, a one-loop diagram in the chiral theory will be proportional to

$$\frac{\Lambda^2}{(4\pi f)^2}$$

where we have included the standard $1/(16\pi^2)$ factor from a loop.

This factor $\Lambda^2/(4\pi f)^2$ controls the size of quantum corrections to operator coefficients in the theory.

If we expect perturbative control of the theory to break down completely for momenta at the cutoff Λ, then we would expect

$$\Lambda \sim 4\pi f$$

Estimating the size of operator coefficients in the chiral Lagrangian by this relation is called "naive dimensional analysis".

Calculation of the ratio $\langle \bar{u}u \rangle / \langle \bar{s}s \rangle$ at one loop

As a simple example of a one-loop calculation, consider the computation of the ratio of the quark condensates:

$$x = \frac{\langle 0|\bar{u}u|0\rangle}{\langle 0|\bar{s}s|0\rangle}$$

The QCD Hamiltonian density is given by

$$\mathcal{H} \supset \bar{q}Mq = \sum_i m_i \bar{q}_i q_i$$

so it follows from the Feynman-Hellman theorem that

$$\langle 0|\bar{q}_i q_i|0\rangle = \frac{\partial}{\partial m_i}\langle 0|\mathcal{H}|0\rangle = \frac{\partial \mathcal{E}_0}{\partial m_i}$$

where $\mathcal{E}_0$ is the vacuum energy density.

We do not know what $\mathcal{E}_0$ is, but we do know its dependence on the quark mass matrix.

From

$$\mathcal{L}_M = \frac{f^2}{2}\tilde{\Lambda}\,\mathrm{tr}(M\Sigma + \mathrm{h.c.}) = \frac{f^2}{2}\tilde{\Lambda}\,\mathrm{tr}[M(\Sigma + \Sigma^\dagger)]$$

it follows that

$$\begin{aligned}\mathcal{E}_0 &\supset \left[-\frac{f^2}{2}\tilde{\Lambda}\,\mathrm{tr}[M(\Sigma + \Sigma^\dagger)] + \mathcal{O}(M^2 \ln M)\right]_{\Sigma_{ij}=\delta_{ij}} \\ &= -f^2\tilde{\Lambda}\,\mathrm{tr}(M) + \mathcal{O}(M^2 \ln M) \\ &= -f^2\tilde{\Lambda}(m_u + m_d + m_s) + \mathcal{O}(M^2 \ln M)\end{aligned}$$

$$\Rightarrow \quad \langle 0|\bar{q}_i q_i|0\rangle = \frac{\partial \mathcal{E}_0}{\partial m_i} = -f^2\tilde{\Lambda} \quad \text{(at leading order)}$$

$$\Rightarrow \quad x = 1 \quad \text{(at leading order)}$$

To get the subleading logarithmic corrections, we need to compute the $\mathcal{O}(m^2 \ln m^2)$ one-loop correction to the vacuum energy.

The vacuum energy density from a real, noninteracting scalar is

$$\mathcal{E}_0 \supset \frac{1}{2} \int \frac{d^4 k_E}{(2\pi)^4} \ln(k_E^2 + m^2)$$

where we have rotated the momentum to the Euclidean space.

Taking $d \equiv 4 - 2\epsilon$, this integral becomes

$$\frac{\mu^{4-d}}{2} \int \frac{d^d k_E}{(2\pi)^d} \ln(k_E^2 + m^2)$$

where the prefactor of μ^{4-d} was included to keep its mass dimension to equal 4.

Proof for $\mathcal{E}_0 \equiv V_{eff} \supset \frac{1}{2}\int \frac{d^4k_E}{(2\pi)^4} \ln(k_E^2 + m^2)$ from a real, free scalar degree of freedom:

A real, noninteracting scalar contributes $e^{\frac{i}{\hbar}\int d^4x(-V_{eff})}$ a factor:

$$e^{\frac{i}{\hbar}\int d^4x(-V_{eff})} \propto \left[\det(\partial^2+m^2)\right]^{-\frac{1}{2}}$$

For a complex scalar, this power is (−1).

$$\Rightarrow \frac{i}{\hbar}\int d^4x(-V_{eff}) \supset \ln\left[\det(\partial^2+m^2)\right]^{-\frac{1}{2}}$$

$$\Rightarrow \int d^4x\, V_{eff} \supset i\hbar \ln\left[\det(\partial^2+m^2)\right]^{-\frac{1}{2}}$$

$$= -\frac{i\hbar}{2}\ln\left[\det(\partial^2+m^2)\right]$$

$$= -\frac{i\hbar}{2}\operatorname{Tr}\ln(\partial^2+m^2)$$

$$= -\frac{i\hbar}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4}\ln(-k^2+m^2)$$

$$\Rightarrow \quad V_{\text{eff}} \supset -\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln(-k^2+m^2)$$

Now make the Wick rotation: $k^0 = ik_4$

$$\Rightarrow \begin{cases} \int d^4k = i\int d^4k_E \\ k^2 = (k^0)^2 - \vec{k}^2 = -k_4^2 - \vec{k}^2 = -k_E^2 \end{cases}$$

$$\Rightarrow \quad V_{\text{eff}} \supset -\frac{i\hbar}{2}\underbrace{\int\frac{d^4k}{(2\pi)^4}}_{= i\int\frac{d^4k_E}{(2\pi)^4}}\ln(\underbrace{-k^2}_{= k_E^2}+m^2) = \frac{\hbar}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln(k_E^2+m^2)$$

✓

This is called the Coleman-Weinberg potential.

Differentiating this integral with respect to the quark mass m_i, we obtain the 1-loop correction to the quark condensate $\langle 0|\bar{q}_i q_i|0\rangle^{(\text{1-loop})}$:

$$
\begin{aligned}
\langle 0|\bar{q}_i q_i|0\rangle^{(\text{1-loop})} &= \frac{\partial \mathcal{E}_0^{(\text{1-loop})}}{\partial m_i} \\
&= \frac{\partial}{\partial m_i} \sum_{a=\pi,K,\eta} \frac{\mu^{4-d}}{2} \int \frac{d^d k_E}{(2\pi)^d} \ln(k_E^2 + m_a^2) \\
&= \sum_a \frac{\mu^{4-d}}{2} \int \frac{d^d k_E}{(2\pi)^d} \frac{\partial m_a^2}{\partial m_i} \frac{\partial}{\partial m_a^2} \ln(k_E^2 + m_a^2) \\
&= \sum_a \frac{\partial m_a^2}{\partial m_i} \frac{\mu^{4-d}}{2} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2 + m_a^2} \\
&\xrightarrow{\overline{\text{MS}}} \sum_a \frac{\partial m_a^2}{\partial m_i} \cdot \frac{m_a^2}{32\pi^2} \ln\left(\frac{m_a^2}{\mu^2}\right)
\end{aligned}
$$

Proof for $\langle 0|\bar{q}_i q_i|0\rangle^{(1\text{-loop})} \xrightarrow{\overline{\text{MS}}} \sum_a \frac{\partial m_a^2}{\partial m_i} \cdot \frac{m_a^2}{32\pi^2} \ln\left(\frac{m_a^2}{\mu^2}\right)$:

In general, $\displaystyle\int \frac{d^d l_E}{(2\pi)^d} \frac{(l_E^2)^n}{(l_E^2+\Delta)^m} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(m-n-\frac{d}{2})\Gamma(n+\frac{d}{2})}{\Gamma(\frac{d}{2})\,\Gamma(m)} \frac{1}{\Delta^{m-n-\frac{d}{2}}}$

$$\overset{(m=1,\ n=0)}{\Longrightarrow} \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{l_E^2+\Delta} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-\frac{d}{2}}}$$

$$\Rightarrow \langle 0|\bar{q}_i q_i|0\rangle^{(1\text{-loop})} = \sum_a \frac{\partial m_a^2}{\partial m_i} \frac{1}{2} \mu^{4-d} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2+m_a^2}$$

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2+m_a^2} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})}{(m_a^2)^{1-\frac{d}{2}}}$$

$$= \sum_a \frac{\partial m_a^2}{\partial m_i} \frac{1}{2} \mu^{4-d} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})}{(m_a^2)^{1-\frac{d}{2}}}$$

Let $d=4-2\epsilon \Rightarrow 2-\frac{d}{2}=\epsilon \Rightarrow \Gamma(1-\frac{d}{2}) = \frac{1}{1-\frac{d}{2}}\Gamma(2-\frac{d}{2}) = \frac{(-1)}{1-\epsilon}\Gamma(\epsilon)$

$$= (-1)[1+\epsilon+O(\epsilon^2)][\tfrac{1}{\epsilon}-\gamma+O(\epsilon)]$$

$$(m_a^2)^{\frac{d}{2}-1} = (m_a^2)^{1-\epsilon} = m_a^2 \cdot (m_a^2)^{-\epsilon} = m_a^2\left[1-\epsilon \ln m_a^2 + O(\epsilon)\right]$$

$$\frac{1}{(4\pi)^{\frac{d}{2}}} = \frac{1}{(4\pi)^2}\left[1+\epsilon \ln(4\pi) + O(\epsilon^2)\right]$$

$$\mu^{4-d} = \mu^{2\epsilon} = e^{\epsilon \ln \mu^2} = 1+\epsilon \ln\mu^2 + O(\epsilon^2)$$

$$\frac{(-1)}{32\pi^2}$$

$$\Rightarrow \langle 0|\bar{q}_i q_i|0\rangle^{(1\text{-loop})} = \sum_a \frac{\partial m_a^2}{\partial m_i}\frac{1}{2}\left[1+\epsilon\ln\mu^2+O(\epsilon^2)\right]\frac{1}{(4\pi)^2}\left[1+\epsilon\ln(4\pi)+O(\epsilon^2)\right]$$

$$\cdot(-1)\left[1+\epsilon+O(\epsilon^2)\right]\left[\frac{1}{\epsilon}-\gamma+O(\epsilon)\right]m_a^2\left[1-\epsilon\ln m_a^2+O(\epsilon^2)\right]$$

$$= \sum_a \frac{\partial m_a^2}{\partial m_i}\frac{(-1)}{32\pi^2}m_a^2\left[\underbrace{\frac{1}{\epsilon}-\gamma+\ln(4\pi)+1}_{\text{removed by }\overline{MS}}-\ln\frac{m_a^2}{\mu^2}+O(\epsilon)\right]$$

$$\xrightarrow{\overline{MS}} \sum_a \frac{\partial m_a^2}{\partial m_i}\cdot\frac{1}{32\pi^2}m_a^2\ln\frac{m_a^2}{\mu^2} \quad \checkmark$$

The quark condensate is

$$\begin{aligned}\langle 0|\bar{q}_i q_i|0\rangle &= \langle 0|\bar{q}_i q_i|0\rangle^{\text{(leading order)}} + \langle 0|\bar{q}_i q_i|0\rangle^{\text{(1-loop)}} \\ &= -f^2\tilde{\Lambda} + \sum_a \frac{\partial m_a^2}{\partial m_i} \cdot \frac{m_a^2}{32\pi^2} \ln\left(\frac{m_a^2}{\mu^2}\right) \\ &= -f^2\tilde{\Lambda}\left[1 - \sum_a \frac{\partial m_a^2}{\partial m_i} \cdot \frac{m_a^2}{32\pi^2 f^2 \tilde{\Lambda}} \ln\left(\frac{m_a^2}{\mu^2}\right)\right]\end{aligned}$$

Thus, the quark condensate ratios are given by

$$\frac{\langle 0|\bar{q}_i q_i|0\rangle}{\langle 0|\bar{q}_j q_j|0\rangle} = 1 - \frac{1}{32\pi^2 f^2 \tilde{\Lambda}} \sum_a \left(\frac{\partial m_a^2}{\partial m_i} - \frac{\partial m_a^2}{\partial m_j}\right) m_a^2 \ln\left(\frac{m_a^2}{\mu^2}\right) + \mathcal{O}\left(\frac{1}{f^4}\right)$$

Using the meson masses and ignoring π^0–η mixing and electromagnetic contributions, we find

$$x \equiv \frac{\langle 0|\bar{u}u|0\rangle}{\langle 0|\bar{s}s|0\rangle} = 1 - 3g_\pi + 2g_{K^0} + g_\eta + \mathcal{O}\left(\frac{1}{f^4}\right)$$

where

$$g_P \equiv \frac{1}{32\pi^2 f^2} m_P^2 \ln\left(\frac{m_P^2}{\mu^2}\right)$$

Real and complex scalar degrees of freedom:

★ Real scalars π^0, η: $\quad m_{\pi^0}^2 \simeq \tilde{\Lambda}(m_u+m_d), \quad m_\eta^2 \simeq \tilde{\Lambda}\,\frac{m_u+m_d+4m_s}{3}$

(ignoring π^0–η mixing)

★ Complex scalars $\pi^\pm, K^\pm, K^0(\bar{K}^0)$: $\quad m_{\pi^\pm}^2 \simeq \tilde{\Lambda}(m_u+m_d), \quad m_{K^0}^2 \simeq \tilde{\Lambda}(m_d+m_s)$

(ignoring electromagnetic contributions) $\quad m_{K^+}^2 \simeq \tilde{\Lambda}(m_u+m_s),$

$$\langle 0|\bar{u}u|0\rangle = -f^2\tilde{\Lambda}\Big[1 - \frac{1}{32\pi^2 f^2\tilde{\Lambda}}\Big(\overset{=\tilde{\Lambda}}{\frac{\partial m_{\pi^0}^2}{\partial m_u}}\cdot m_{\pi^0}^2 \ln\frac{m_{\pi^0}^2}{\mu^2} + \overset{=\frac{\tilde{\Lambda}}{3}}{\frac{\partial m_\eta^2}{\partial m_u}}\cdot m_\eta^2 \ln\frac{m_\eta^2}{\mu^2}$$

$$+2\overset{=\tilde{\Lambda}}{\frac{\partial m_{\pi^+}^2}{\partial m_u}}\cdot m_{\pi^+}^2 \ln\frac{m_{\pi^+}^2}{\mu^2} + 2\overset{=\tilde{\Lambda}}{\frac{\partial m_{K^+}^2}{\partial m_u}}\cdot m_{K^+}^2 \ln\frac{m_{K^+}^2}{\mu^2}\Big)\Big]$$

$$= -f^2\tilde{\Lambda}\Big[1 - \frac{1}{32\pi^2 f^2}\Big(\underbrace{(1+2)}_{=3} m_\pi^2 \ln\frac{m_\pi^2}{\mu^2} + \frac{1}{3}m_\eta^2 \ln\frac{m_\eta^2}{\mu^2} + 2m_{K^+}^2 \ln\frac{m_{K^+}^2}{\mu^2}\Big)\Big]$$

$$\langle 0|\bar{s}s|0\rangle = -f^2\tilde{\Lambda}\Big[1 - \frac{1}{32\pi^2 f^2\tilde{\Lambda}}\Big(\overset{=\frac{4}{3}\tilde{\Lambda}}{\frac{\partial m_\eta^2}{\partial m_s}}\cdot m_\eta^2 \ln\frac{m_\eta^2}{\mu^2} + 2\overset{=\tilde{\Lambda}}{\frac{\partial m_{K^+}^2}{\partial m_s}}\cdot m_{K^+}^2 \ln\frac{m_{K^+}^2}{\mu^2} + 2\overset{=\tilde{\Lambda}}{\frac{\partial m_{K^0}^2}{\partial m_s}}\cdot m_{K^0}^2 \ln\frac{m_{K^0}^2}{\mu^2}\Big)\Big]$$

$$= -f^2\tilde{\Lambda}\Big[1 - \frac{1}{32\pi^2 f^2}\Big(\frac{4}{3}m_\eta^2 \ln\frac{m_\eta^2}{\mu^2} + 2m_{K^+}^2 \ln\frac{m_{K^+}^2}{\mu^2} + 2m_{K^0}^2 \ln\frac{m_{K^0}^2}{\mu^2}\Big)\Big]$$

$$\Rightarrow \frac{\langle 0|\bar{u}u|0\rangle}{\langle 0|\bar{s}s|0\rangle} \simeq 1 - \frac{1}{32\pi^2 f^2}\Big(3m_\pi^2 \ln\frac{m_\pi^2}{\mu^2} + \frac{1}{3}m_\eta^2 \ln\frac{m_\eta^2}{\mu^2} \cancel{+ 2m_{K^+}^2 \ln\frac{m_{K^+}^2}{\mu^2}}$$

$$- \frac{4}{3}m_\eta^2 \ln\frac{m_\eta^2}{\mu^2} \cancel{- 2m_{K^+}^2 \ln\frac{m_{K^+}^2}{\mu^2}} - 2m_{K^0}^2 \ln\frac{m_{K^0}^2}{\mu^2}\Big)$$

$$= 1 - \frac{1}{32\pi^2 f^2}\Big(3m_\pi^2 \ln\frac{m_\pi^2}{\mu^2} - m_\eta^2 \ln\frac{m_\eta^2}{\mu^2} - 2m_{K^0}^2 \ln\frac{m_{K^0}^2}{\mu^2}\Big)$$

$$= 1 - 3g_\pi + g_\eta + 2g_{K^0} \qquad \checkmark$$

where $\quad g_P \equiv \frac{1}{32\pi^2 f^2} m_P^2 \ln\frac{m_P^2}{\mu^2}$

The ratio $x \equiv \frac{\langle 0|\bar{u}u|0\rangle}{\langle 0|\bar{s}s|0\rangle} \simeq 1 - 3g_\pi + 2g_{K^0} + g_\eta$ with
$g_P \equiv \frac{1}{32\pi^2 f^2} m_P^2 \ln\left(\frac{m_P^2}{\mu^2}\right)$ is μ dependent.

We can take $\mu = \Lambda = 1$ GeV and assume the contributions from the $\mathcal{O}(p^4)$ Lagrangian are small compared to the chiral logs we have included.

Plugging in numbers we find

$$g_\pi \simeq -0.028, \quad g_{K^0} \simeq -0.13, \quad g_\eta \simeq -0.13$$

$$\Rightarrow \quad x \simeq 0.70$$

which is a 30% correction from the leading order result $x = 1$.

```
In[1]:= (*Define the meson masses in unit of MeV from the webpage: https://
         pdg.lbl.gov/2021/tables/rpp2021-sum-mesons.pdf *)
        MassPiPlus = 139.57039; MassPi0 = 134.9768; MassEta = 547.862; MassKPlus = 493.677;
        MassK0 = 497.611; gp[f_, Λ_, MesonMass_] = 1/(32 π^2 f^2) MesonMass^2 Log[MesonMass^2/Λ^2]
```

$$\text{Out[1]=}\quad \frac{\text{MesonMass}^2\,\text{Log}\left[\frac{\text{MesonMass}^2}{\Lambda^2}\right]}{32\,f^2\,\pi^2}$$

```
In[2]:= gp[92.4, 1000, MassPi0]
        gp[92.4, 1000, MassPiPlus]
Out[2]= -0.027062
```

g_{π^0}

```
Out[3]= -0.0284518
```

g_{π^+}

```
In[4]:= gp[92.4, 1000, MassK0]
Out[4]= -0.128184
```

g_{K^0}

```
In[5]:= gp[92.4, 1000, MassEta]
Out[5]= -0.133962
```

g_η

$$x = 1 - 3g_{\pi^+} + 2g_{K^0} + g_\eta$$

```
In[6]:= x = 1 - 3 gp[92.4, 1000, MassPiPlus] + 2 gp[92.4, 1000, MassK0] + gp[92.4, 1000, MassEta]
Out[6]= 0.695026

In[7]:= x2 = 1 - gp[92.4, 1000, MassPi0] - 2 gp[92.4, 1000, MassPiPlus] + 2 gp[92.4, 1000, MassK0] +
          gp[92.4, 1000, MassEta]
Out[7]= 0.693637
```

$$x = 1 - g_{\pi^0} - 2g_{\pi^+} + 2g_{K^0} + g_\eta$$

Summary and Outlook

♣ Chiral Lagrangian is a useful tool to describe the physics of Goldstone or Pseudo-Goldstone bosons, for example QCD pions.

♣ Chiral Lagrangian can be used to discuss some BSM theories, such as Composite Higgs and Technicolor theories.

♣ Other applications include: BEC of pions in neutron stars; dense superconducting quark matter, etc.

Main reference:
David B. Kaplan, Lecture on effective field theory (2016)