Cosmology: Introduction

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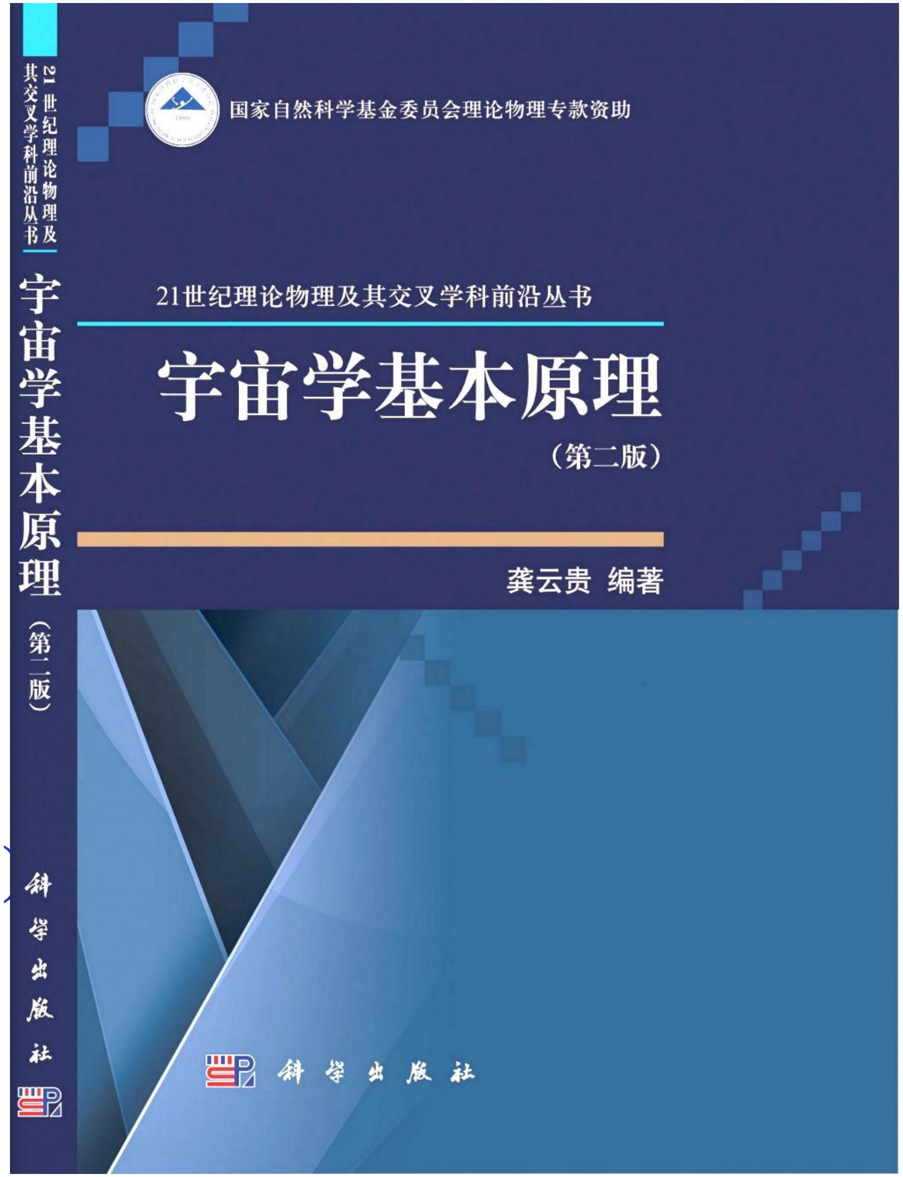
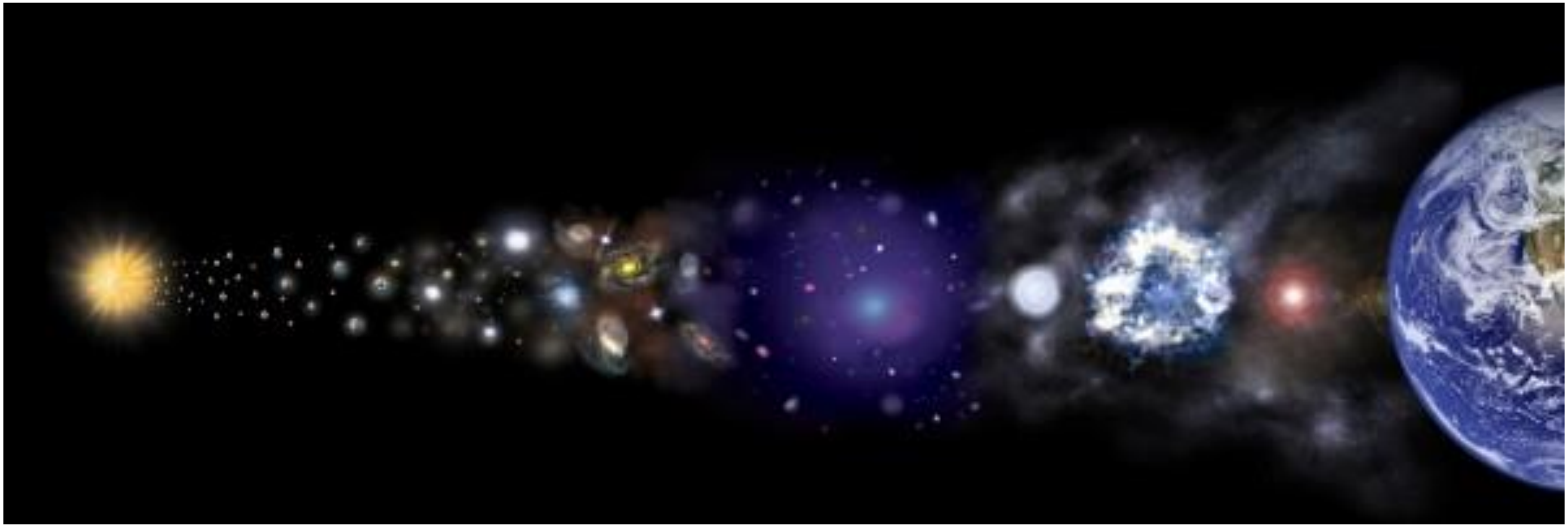


Outline

- Cosmological Evolution
- Thermal history
- Inflation (accelerating expansion)
- Cosmic microwave background radiation (CMBR)
- Cosmological parameters (Hubble tension)

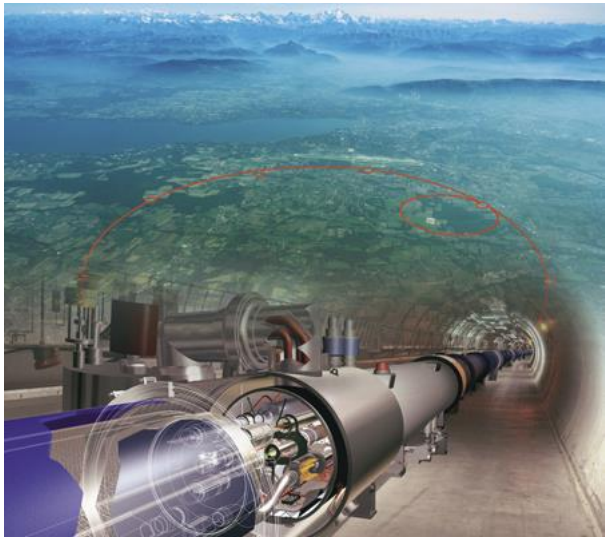
The Universe

- Universe (宇宙): Two words in Chinese
- The first word means all spaces around us
- The second word means the whole time
- Universe：All spaces and the whole time



New Physics Lab

- The early Universe (inflation): natural lab for particle physics $\lesssim 10^{18}\ \mathrm{GeV}$
- LHC： $14\ \mathrm{TeV}$
- The origin of the Universe： Quantum Gravity？GUT？ $\frac{1}{G} \sim 10^{19}\ \mathrm{GeV}$



Cosmology

- Cosmology: study the whole universe
- Fundamental forces: only gravity and electromagnetic forces are long range forces
- Gravity: Einstein's general relativity
- Cosmological principle: Space-time is Isotropic and Homogeneous at large scales

14000Mpc~10^{26} m
Galaxy size: a few Mpc

General Relativity

- Einstein's equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} R_{\mu\nu}$

Line element (metric) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Christoffel connection $\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$

Riemann curvature tensor $R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\alpha_{\mu\rho} \Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\nu\rho} \Gamma^\sigma_{\alpha\mu}$

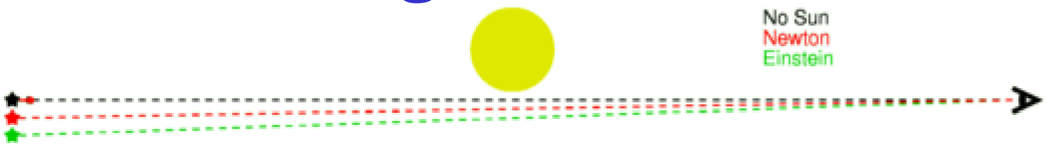
Ricci tensor $R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha}$

Ricci scalar $R = g^{\mu\nu} R_{\mu\nu} = R_\mu{}^\mu$

Covariant derivative $A^\mu{}_{;\nu} = A^\mu{}_{,\nu} + \Gamma^\mu_{\nu\alpha} A^\alpha \quad A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} A_\alpha$

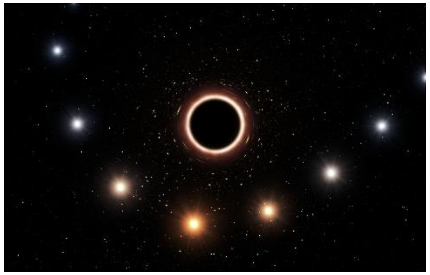
Test on GR

- Light deflection（gravitational lensing）



Result: 0.9998 ± 0.0008 ， Lebach et al. 1995, PRL, **75**, 1439

Shapiro time delay: 1.00001 ± 0.000012
Bertotti, Iess & Tortora 2003, Nature, **425**, 374-376



- Gravitational redshift

7×10^{-5}

R.F.C. Vessot etal. 1980, PRL **45**, 2081

2.5×10^{-5}

P. Delva etal. 2019, Comptes Rendus Physique **20,** 176-182

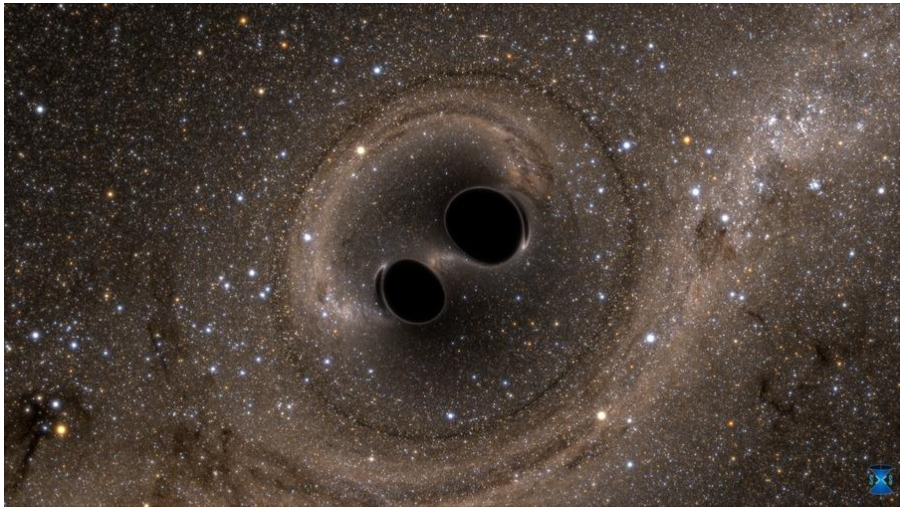
$0.90 \pm 0.09|_{\rm stat} \pm 0.15|_{\rm sys}$

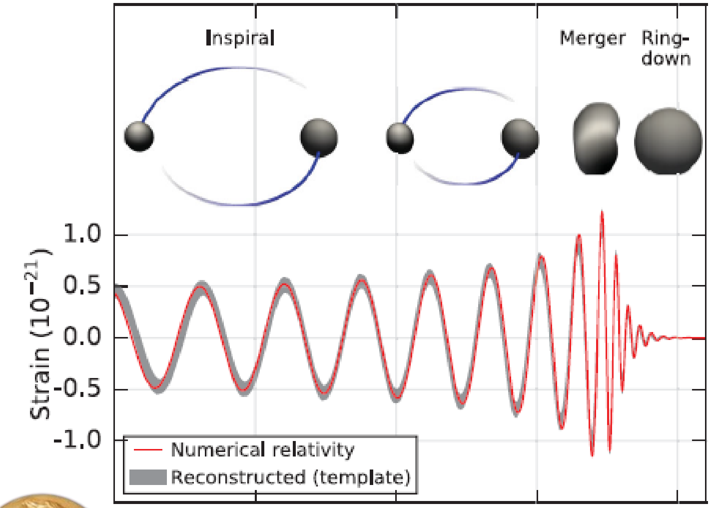
GRAVITY Collaboration 2018, A&A **615**, L15

Other solar system test: The precession of perihelion of Mercury

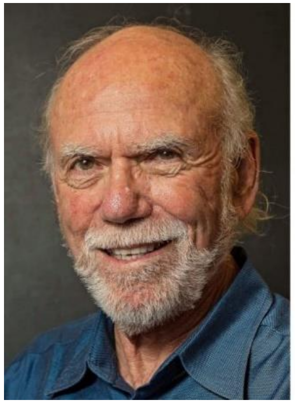
Gravitational waves

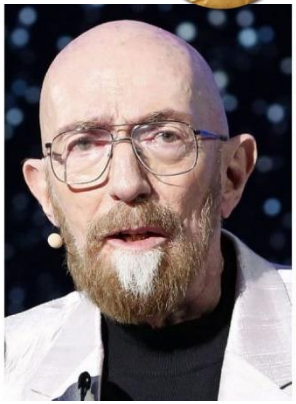
The discovery of GWs

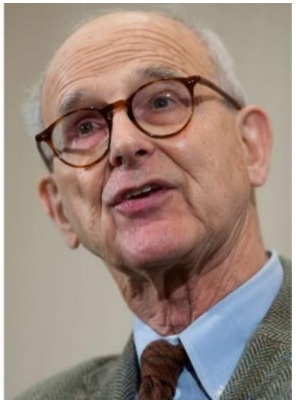




2017 Nobel Prize


Barry C. Barish


Kip S. Thorne


Rainer Weiss

Cosmological principle

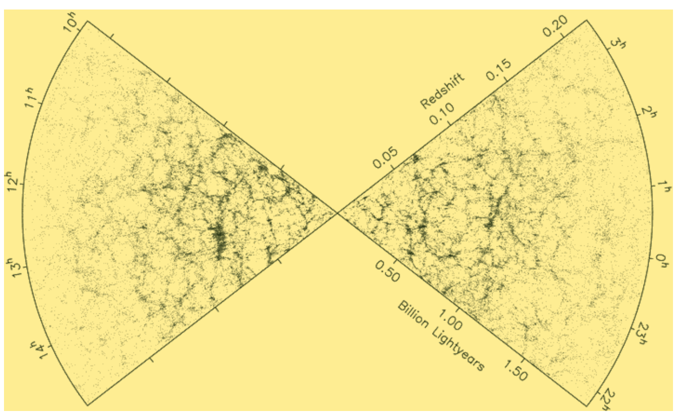
- Cosmological principle: Space-time is Isotropic and Homogeneous at large scales

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

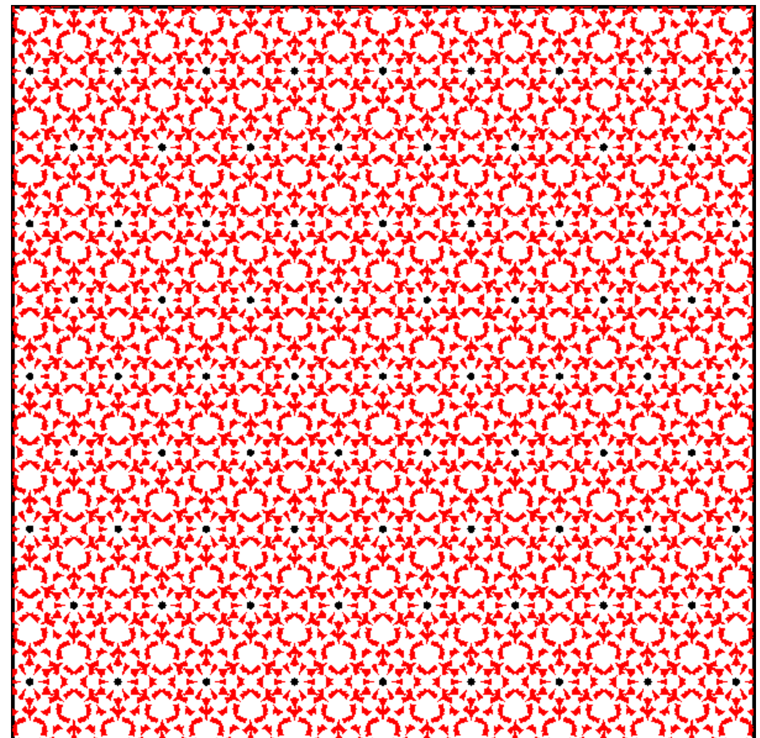
Robertson-Walker metric

14000Mpc~10^{26} m

Galaxy size: a few Mpc



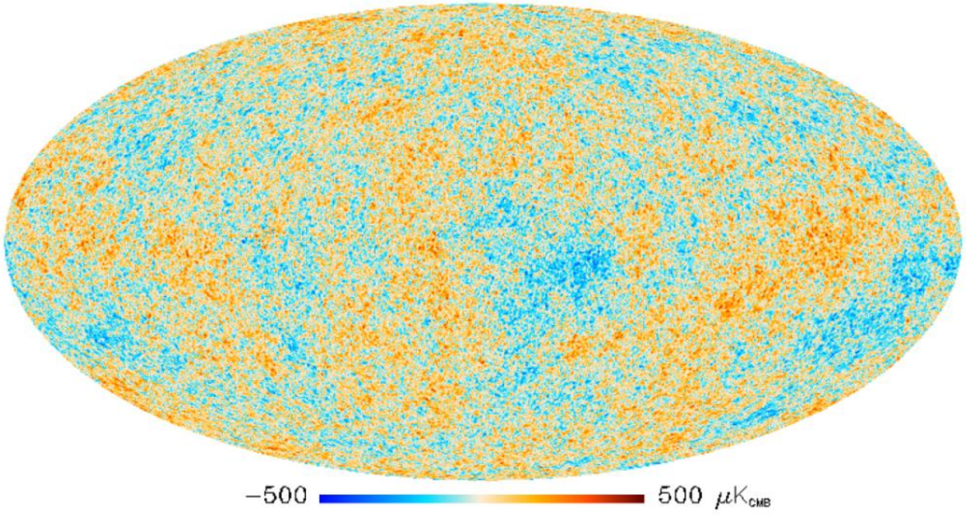
Structure at smaller scales



12.1 Gyr, photons from all galaxies shown

Cosmological Principle

$$T = 2.72548 \pm 0.00057\ ^{\circ}\mathrm{K}$$



Planck 2018, 1807.06205, *AA* 641 (2020) A1

Universe 380,000 years old,13.7 billion years ago

Energy Conservation

- Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

- Energy-momentum tensor

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) U_\mu U_\nu$$

$$T^{\mu\nu}_{;\nu} = 0 \longrightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \qquad p = f(\rho)$$

$$w = p/\rho, \quad \rho \propto a^{-3(1+w)}$$

Equation of state

$$w = \frac{p}{\rho}$$

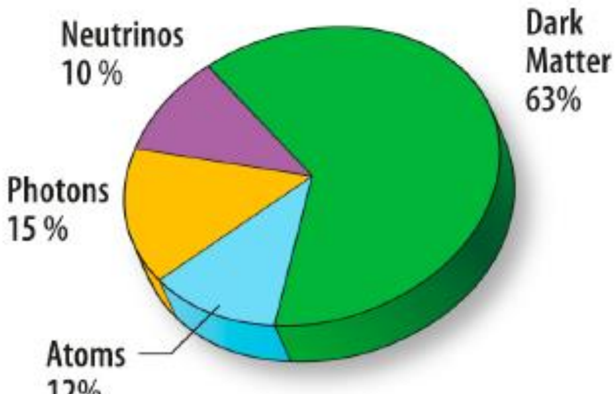
- Dust

$$w = 0, \quad \rho \propto a^{-3}$$

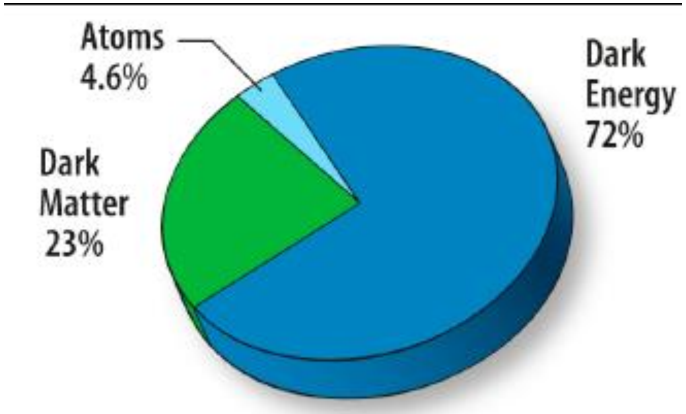
- Radiation

$$w = 1/3, \quad \rho \propto a^{-4} \propto T_{CMB}^4$$

More general $\rho \propto a^{-3(1+w)}$



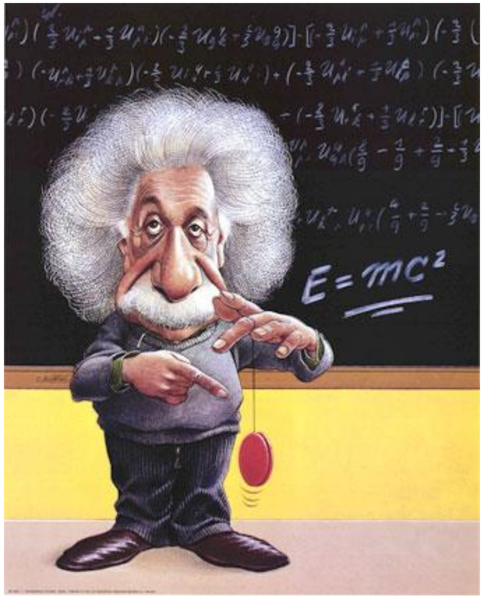
13.7 BILLION YEARS AGO
(Universe 380,000 years old)

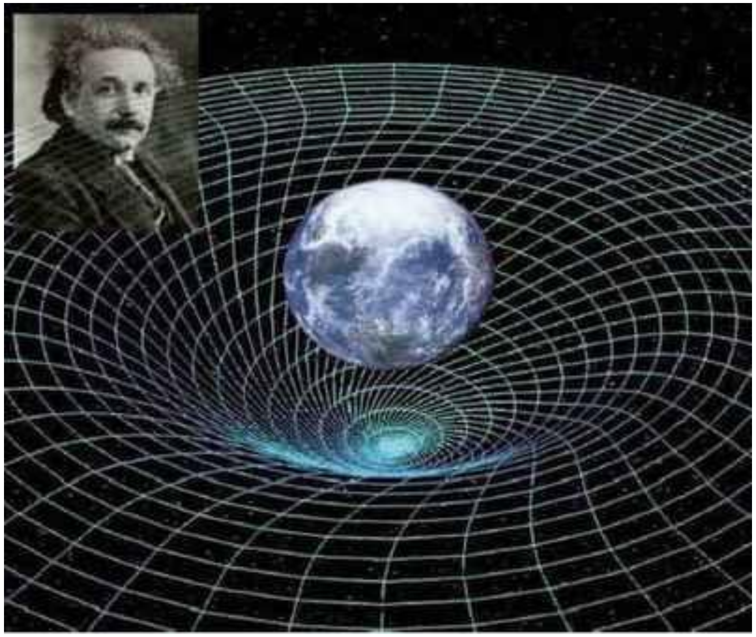


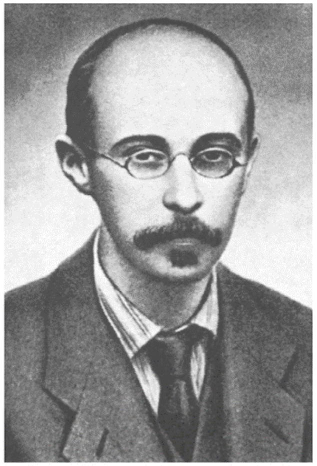
TODAY

Standard cosmology

- Einstein's general relativity $G_{\mu\nu} = 8\pi G T_{\mu\nu}$
- Friedmann Equation $$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho}$$

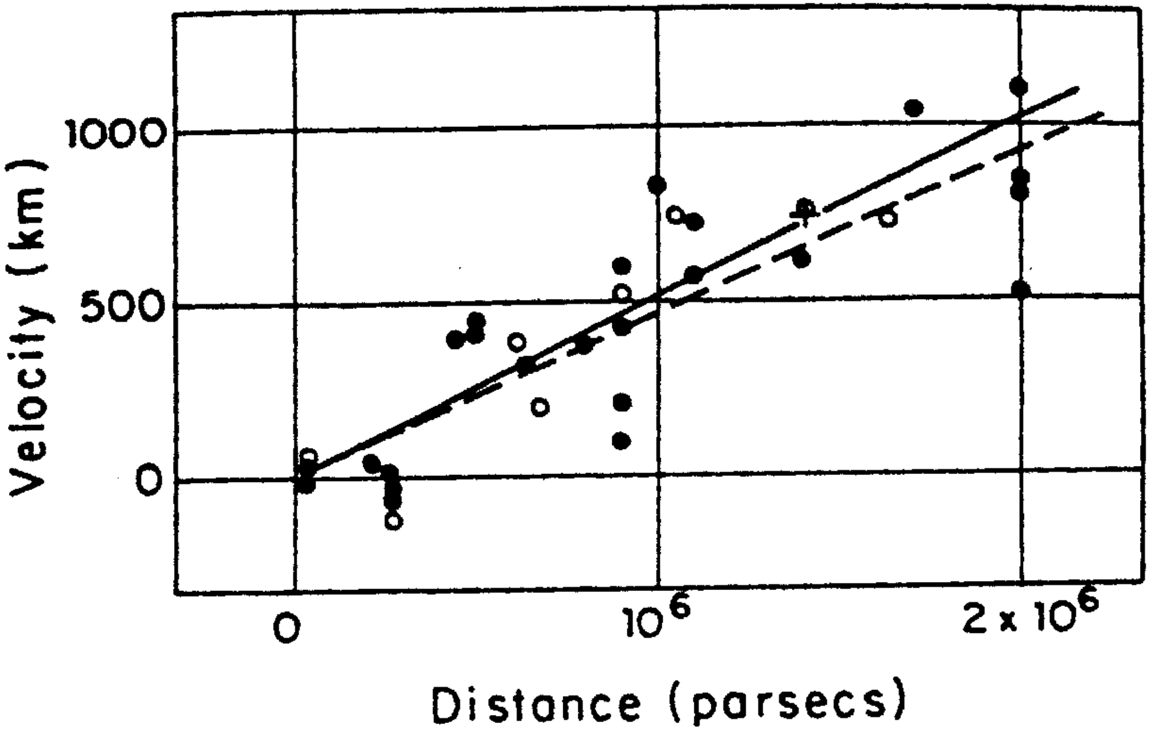






Hubble's Law

- Expanding Universe

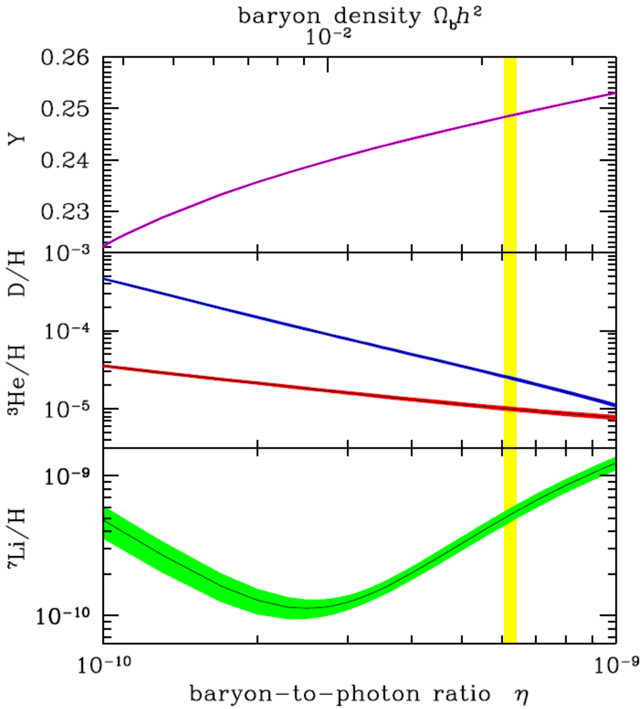


1929

Hubble

Big Bang Cosmology

- Big Bang (Hoyle 1949)
- Prediction of CMB (Gamow): confirmed in 1965
- Explanation of the primordial abundances of elements
- Thermal history
- Large scale structure
- …..



Cosmological equations

- Friedmann Eq.

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

$$H(t) = \frac{\dot{a}}{a}$$

- Acceleration

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

- Energy conservation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \qquad p = f(\rho)$$

- Deceleration parameter

$$q(t) \equiv -\frac{\ddot{a}}{aH^2} = -\frac{1}{aH^2}\frac{d^2a}{dt^2}$$

Cosmological evolution

- Einstein-de Sitter universe $K=0, \quad \Omega_k = 0$

$$w = p/\rho = 0$$

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2\left(\frac{a_0}{a}\right) \qquad a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3t} \qquad \rho_m = \rho_{m0}\left(\frac{t}{t_0}\right)^{-2} = \frac{1}{6\pi G t^2}$$

- Radiation domination

$$w = p/\rho = 1/3 \quad \rho_r = \rho_{r0}\left(\frac{a_0}{a}\right)^4 = \frac{3}{32\pi G t^2}$$

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{1/2} \qquad H(t) = \frac{\dot{a}}{a} = \frac{1}{2t}$$

Cosmological constant

- Einstein's equation with cosmological constant

Geometry $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$

Cosmological constant $G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$

$$T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) U_\mu U_\nu$$

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G} \qquad w = p/\rho = -1$$

- de-Sitter Universe

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \qquad a \propto \exp(Ht)$$

Accelerating Universe $\dfrac{\ddot{a}}{a} = -\dfrac{4\pi G}{3}(\rho + 3p)$ Repulsive force

Cosmological constant problem

- Vacuum energy $$E_0 = \frac{1}{2}\sum_j \hbar\omega_j$$

$$E_0 = \frac{1}{2}\hbar L^3 \int \frac{d^3k}{(2\pi)^3}\omega_k$$

$$\omega_k^2 = k^2 + m^2/\hbar^2, \quad k_{max} \gg m/\hbar$$

$$\rho_{vac} = \lim_{L\to\infty}\frac{E_0}{L^3} = \hbar\frac{k_{max}^4}{16\pi^2}$$

$$k_{max} = (E_{planck} \approx 10^{19}\ \mathrm{Gev})/\hbar$$

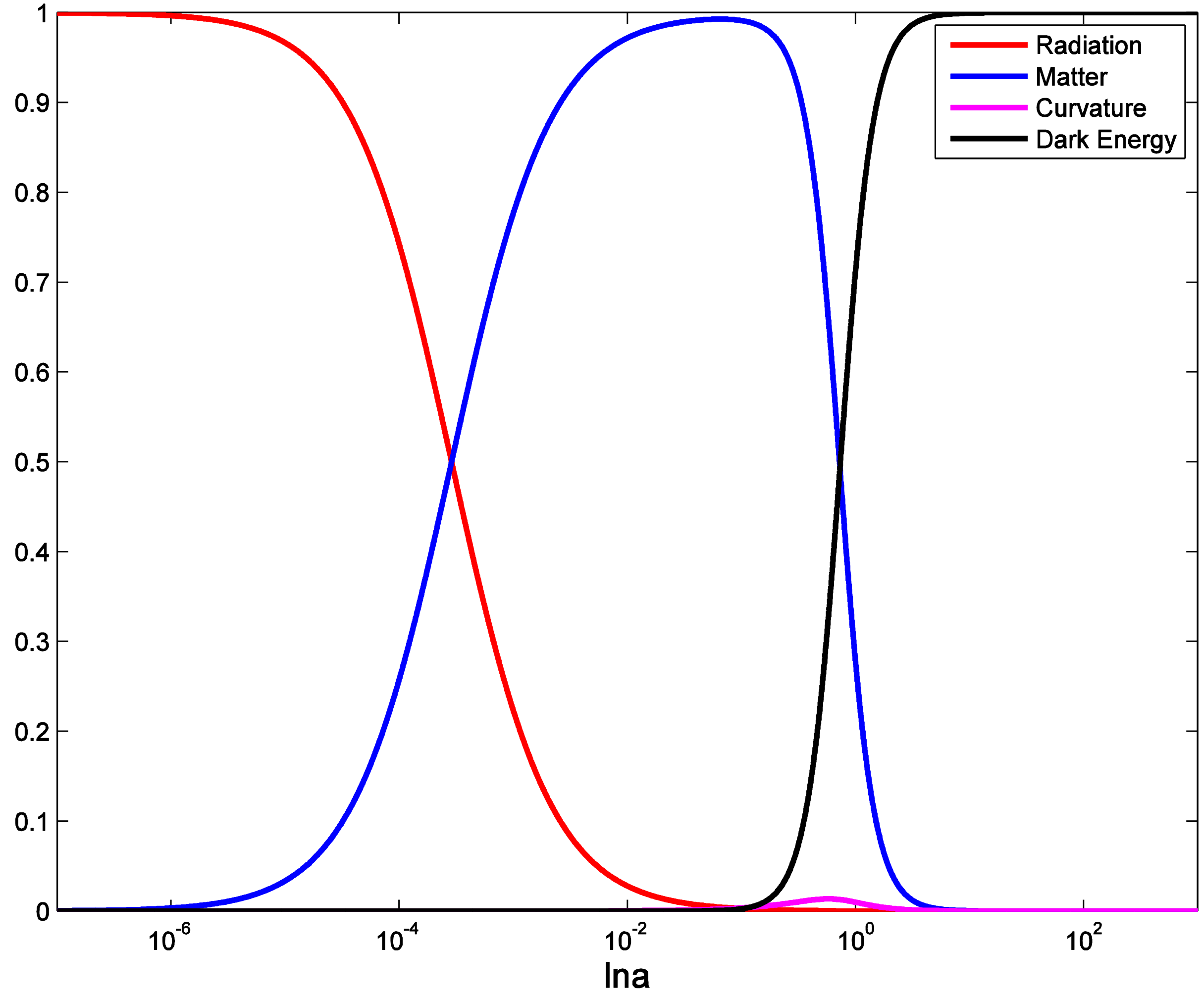
$$\rho_{vac} = 10^{74}\mathrm{Gev}^4/\hbar^3 \approx 10^{92}\mathrm{g/cm}^3 = 10^{120}\rho_\Lambda$$

Accelerating Expansion

- The discovery of the accelerating expansion of the Universe through observations of distant supernovae

2011 Nobel Prize

The evolution of different matter



Thermal History

- Entropy $$s = \frac{\rho(T) + p(T)}{T} = \frac{2\pi^2}{45} g_{*S}(T) T^3$$

$$g_{*S}(T) = \sum_{i=bosons} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=fermions} g_i \left(\frac{T_i}{T}\right)^3$$

- Radiation $\rho \propto T^4 \propto a^{-4}$

Before the decoupling of neutrinos

$$T = 10^{11}\ {}^\circ\mathrm{K} \qquad m_\mu \gg k_B T \gg m_e$$

宇宙中含有光子（有效自由度 $g_\gamma = 2$ ），正负电子（有效自由度 $g_e = 4$ ），三代中微子及反中微子这些处于热平衡的高度相对论性粒子。由于中微子只有一个自旋态，而且正反中微子有**3**代，中微子自由度为 $g_\nu = 6$

$$g_* = 2 + \frac{7}{8}(6+4) = \frac{43}{4}$$

Neutrino temperature

- 电子退耦　　当宇宙温度低于**0.5Mev**，电子退耦

退耦前　$T > m_e,\ g_* = g_{*S} = 2 + 7/8 \times 4 = 11/2$

退耦后　$T < m_e,\ g_* = g_{*S} = 2$

电子对的湮灭使得光子的温度增加了一个因子

$$\left(\frac{g_{*S}(T > m_e)}{g_{*S}(T < m_e)}\right)^{1/3} = \left(\frac{11/2}{2}\right)^{1/3} = \left(\frac{11}{4}\right)^{1/3}$$

中微子已经退耦，其温度 $T_\nu \propto a^{-1}(t)$

$T < m_e$ 后到现在光子的温度和中微子的温度比值为

$$\frac{T_\gamma}{T_\nu} = \left(\frac{11}{4}\right)^{1/3} = 1.40$$　CMB

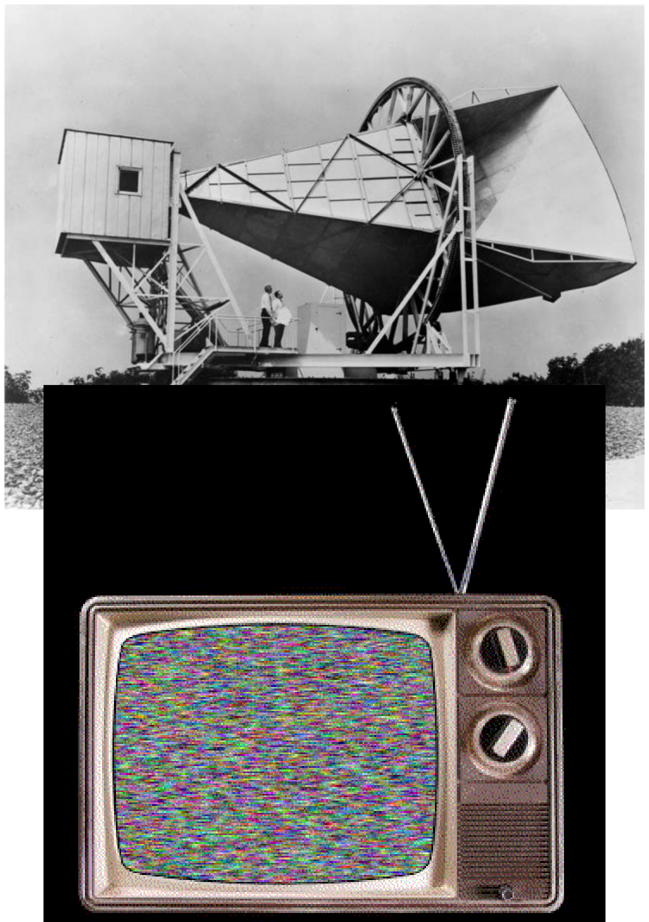
Thermal history

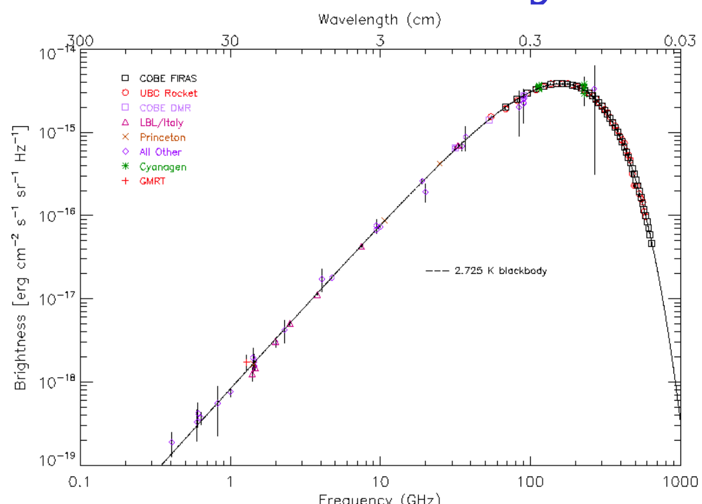
	时间	能量	
普朗克时代?	$< 10^{-43}$ s	10^{18} GeV	
超弦?	$\gtrsim 10^{-43}$ s	$\lesssim 10^{18}$ GeV	
大统一?	$\sim 10^{-36}$ s	10^{15} GeV	
暴涨?	$\gtrsim 10^{-34}$ s	$\lesssim 10^{15}$ GeV	
超对称破缺?	$< 10^{-10}$ s	> 1 TeV	
重子形成?	$< 10^{-10}$ s	> 1 TeV	
弱电统一	10^{-10} s	1 TeV	
夸克-强子转变	10^{-4} s	10^2 MeV	
核子冻结	0.01 s	10 MeV	
中微子退耦	1 s	1 MeV	
BBN	3 min	0.1 MeV	
			红移
物质-辐射相等	10^4 yrs	1 eV	10^4
重新组合	10^5 yrs	0.1 eV	1,100
暗世纪	$10^5 - 10^8$ yrs		> 25
重新电离	10^8 yrs		$25 - 6$
星系形成	$\sim 6 \times 10^8$ yrs		~ 10
暗能量	$\sim 10^9$ yrs		~ 2
太阳系	8×10^9 yrs		0.5
现在	13.7×10^9 yrs	1 meV	0

$$\frac{T_\gamma}{T_\nu} = \left(\frac{11}{4}\right)^{1/3} = 1.40$$

CMB

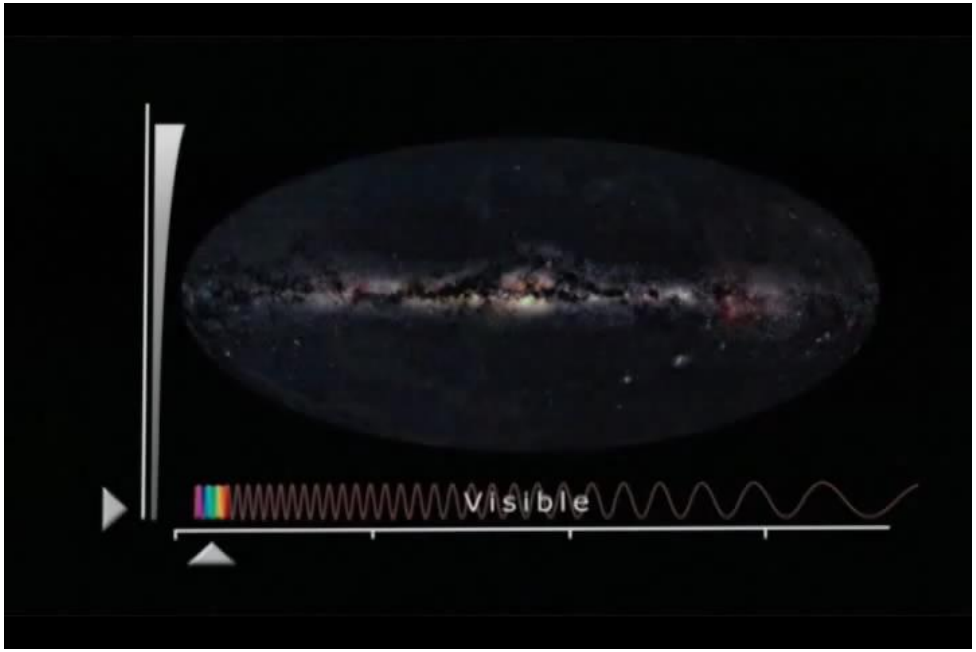
- CMB（1965）： test cosmological principle
- Nobel prize（1978）： The discovery of CMB

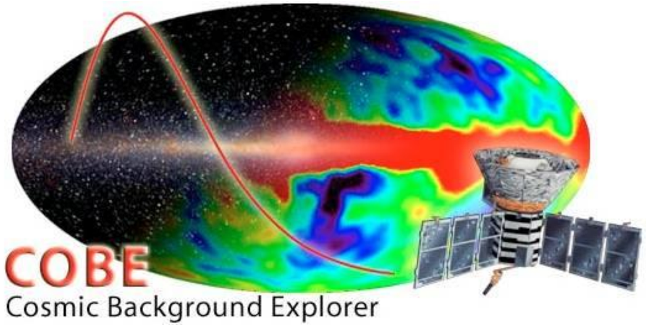


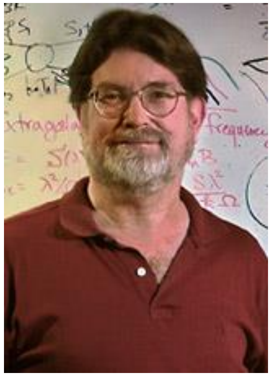


CBM anisotropy

- Anisotropy： COBE 1991， its origin？
- Noble prize（2006）： discovery of anisotropy





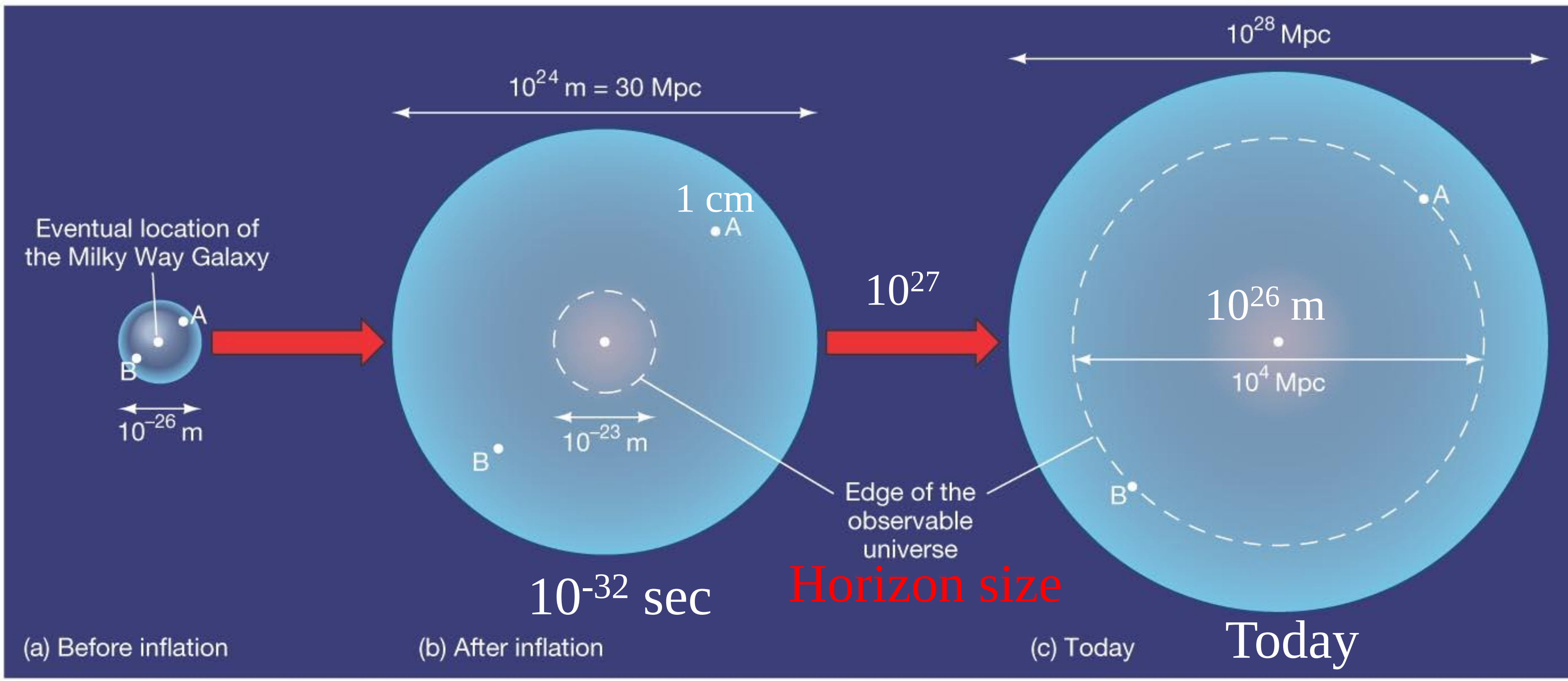


George F. Smoot **John C. Mather**

Horizon Problems

- The expansion speed is less than light speed

$$t = 10^{-32}\mathrm{s},\ 2ct = 10^{-23}\mathrm{m}$$
$$t_0 = 10^{18}\mathrm{s},\ 2ct_0 = 10^{27}\mathrm{m} = 10^4\mathrm{Mpc}$$
$$\frac{a_0}{a} = (1+z_{eq})^{1/4}\left(\frac{t_0}{t}\right)^{1/2} = 10^{27}$$



Flatness problem

- Why $\Omega_K \approx 0$ at the beginning

Curvature density

$$\Omega_K(z) = -K/(a^2H^2) = \Omega_{K0}(1+z)^2/E^2(z)$$

MD

$$E^2(z) \simeq (1+z)^3, \quad \Omega_K(z) \simeq \Omega_{K0}/(1+z)$$

Matter-Radiation Equality $\Omega_K \sim 10^{-4}\Omega_{K0}$

RD

$$E^2(z) \sim (1+z)^4, \quad \Omega_K(z) \sim \Omega_{K0}/(1+z)^2$$

Problems

- Flatness Problem: Why does the universe appear so flat? not clearly open or closed
- Relics Problem: Why do we see no monopoles?
- Horizon Problem

The Universe looks the same everywhere in the sky that we look? The entire universe must have been at uniform temperature near beginning, There has not been enough time since the big bang for light to travel between two parts on opposite horizons

- Dark energy: repulsive force
- Dark matter

Inflation Theory

- Guth (1980s): The Universe expanded exponentially fast at very early time

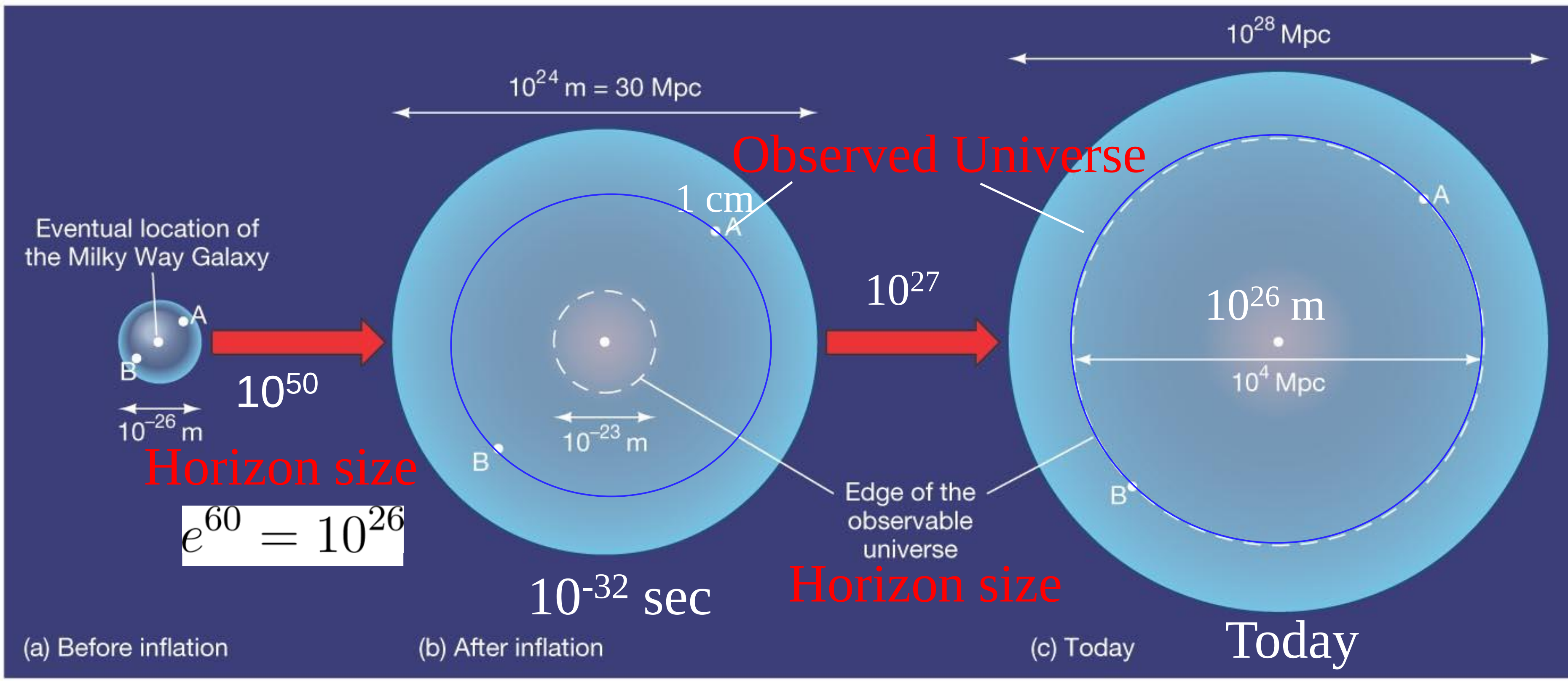
Horizon Problems

- The expansion speed is less than light speed

$$t = 10^{-32}\mathrm{s},\ 2ct = 10^{-23}\mathrm{m}$$

$$t_0 = 10^{18}\mathrm{s},\ 2ct_0 = 10^{27}\mathrm{m} = 10^4\mathrm{Mpc}$$

$$\frac{a_0}{a} = (1+z_{eq})^{1/4}\left(\frac{t_0}{t}\right)^{1/2} = 10^{27}$$



Inflationary Solution

- Flatness: Inflation pushes the Universe towards flatness (stretch away any unevenness)

Longer inflation ⟶ Flatter Universe

- Relics: Inflation greatly dilutes any relics

⟶ We should not observe them today

Inflation

- Conditions

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\rho + 3p < 0$$

$$\ddot{a} > 0 \Longleftrightarrow \frac{d}{dt}\left(\frac{1}{aH}\right) < 0$$

- Inflation is equivalent to the decrease in co-moving Hubble horizon, and it can solve the problems in the standard cosmology

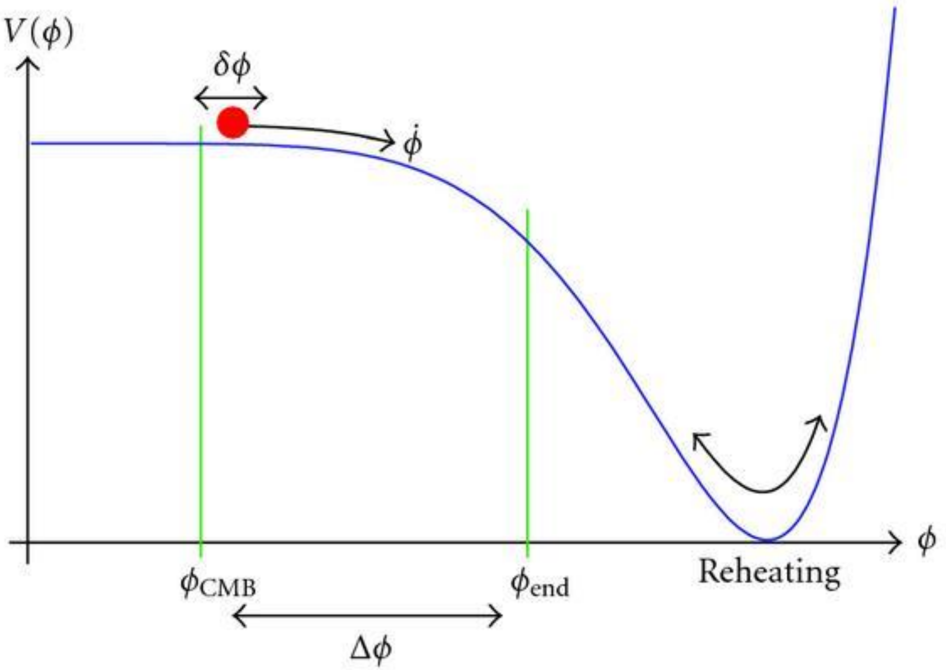
Inflationary Models

- Inflation: accelerated expansion, repulsive force
- Scalar field: if potential energy is bigger than kinetic energy, drives accelerated expansion
- Flat potential:

to get enough inflation

slow-roll inflation

slow-roll parameters



Second order approximation

- The action (to the second order)

$$\delta_2 S = \frac{1}{2}\int dt d^3x \frac{\dot\phi^2}{H^2}\left[a^3\dot\zeta^2 - a(\zeta_{,i})^2\right]$$
$$= \frac{1}{2}\int d\tau d^3x \frac{a^2\phi'^2}{\mathscr{H}^2}\left[\zeta'^2 - (\zeta_{,i})^2\right]$$
$$= \frac{1}{2}\int d\tau d^3x \left[v'^2 - (v_{,i})^2 + \frac{z''}{z}v^2\right],$$

Simple harmonic oscillators

$$v = a\phi'\zeta/\mathscr{H},\ \phi' = d\phi/d\tau,\ \mathscr{H} = d\ln a/d\tau$$

$$z = \frac{a\phi_0'}{\mathscr{H}}$$

$$\delta_2 S = \frac{1}{2}\int\left(v'^2 - \gamma^{ij}v_{,i}v_{,j} + \frac{z''}{z}v^2\right)d^3x d\tau,\quad v' = dv/d\tau$$

$$v = -z\mathscr{R}$$

Quantization

- Canonical quantization

Conjugate momentum $\pi(\tau,\vec{x}) = \delta L/\delta v' = v'(\tau,\vec{x})$

Hamiltonian

$$H = \int (v'\pi - L)\sqrt{\gamma}d^3x = \frac{1}{2}\int \left(\pi^2 + \gamma^{ij}v_{,i}v_{,j} - \frac{z''}{z}v^2\right)\sqrt{\gamma}d^3x$$

$$[\hat{v}(\tau,\vec{x}),\ \hat{v}(\tau,\vec{x}')] = [\hat{\pi}(\tau,\vec{x}),\ \hat{\pi}(\tau,\vec{x}')] = 0,$$

$$[\hat{v}(\tau,\vec{x}),\ \hat{\pi}(\tau,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}'),$$

$$\int \delta^{(3)}(x-x')\sqrt{\gamma}d^3x = 1$$

$$i\hat{v}' = [\hat{v},\ \hat{H}], \quad i\hat{\pi}' = [\hat{\pi},\ \hat{H}]$$

Quantization

- quantization

$$\hat{v}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}[v_k(\tau)a_k e^{i\vec{k}\cdot\vec{x}} + v_k^*(\tau)a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}]$$

$$[\hat{v}(\tau,\vec{x}),\ \hat{v}(\tau,\vec{x}')] = [\hat{\pi}(\tau,\vec{x}),\ \hat{\pi}(\tau,\vec{x}')] = 0,$$

$$[\hat{v}(\tau,\vec{x}),\ \hat{\pi}(\tau,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}'),$$

$$[a_k,\ a_{k'}] = [a_k^\dagger,\ a_{k'}^\dagger] = 0, \quad [a_k,\ a_{k'}^\dagger] = \delta^{(3)}(\vec{k}-\vec{k}')$$

- Bunch-Davies vacuum $a_k|0\rangle = 0$

$$v_k^*\frac{dv_k}{d\tau} - v_k\frac{dv_k^*}{d\tau} = -i$$

Mukhanov-Sassaki Eq.

- Mode function $\boxed{v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0}$

Asymptotic solution

- EOM

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$$

- Well inside the horizon

$$v_k(\tau) \to \frac{1}{2k} e^{-ik\tau}, \quad k \to \infty \qquad\qquad v_k'' + k^2 v_k \approx 0$$

$$v_k^* \frac{dv_k}{d\tau} - v_k \frac{dv_k^*}{d\tau} = -i$$

- Superhorizon

$$v_k(\tau) \propto z, \quad k \to 0 \qquad\qquad v_k'' - \frac{z''}{z} v_k = 0$$

Comoving curvature perturbation $\mathscr{R} = \frac{v}{z}$ is a constant

Quantum fluctuation

- First order

$$v_k'' + \left(k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right) v_k = 0 \qquad \nu = 3/2 + 2\epsilon_H - \eta_H \approx \text{常数}$$

$$v_k(\tau) = \sqrt{-\tau}[c_1(k)H_\nu^{(1)}(-k\tau) + c_2(k)H_\nu^{(2)}(-k\tau)]$$

$$c_2(k) = 0$$

Boundary condition $$v_k(\tau) = \frac{\sqrt{\pi}}{2}e^{i(\nu+1/2)\pi/2}\sqrt{-\tau}H_\nu^{(1)}(-k\tau)$$

- Co-moving curvature perturbation on super-horizon

$$|\mathscr{R}_k| = \left|\frac{v_k}{z}\right| = \left|\frac{H}{\dot{\phi}_0}\frac{v_k}{a}\right| = \frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{H}{\dot{\phi}_0}\frac{H}{\sqrt{2k^3}}\left(\frac{k}{2aH}\right)^{3/2-\nu}, \quad k < aH.$$

Quantum fluctuation

- Power spectrum

$$\hat{v}_k = v_k a_k + v_k^* a_k^\dagger$$

$$\boxed{\langle \hat{v}_{k_1} \hat{v}_{k_2}^* \rangle = v_{k_1} v_{k_2}^* \langle 0 | a_{k_1} a_{k_2}^\dagger | 0 \rangle} = v_{k_1} v_{k_2}^* \langle 0 | [a_{k_1}, \ a_{k_2}^\dagger] | 0 \rangle$$

$$= |v_{k_1}|^2 \delta^{(3)}(\vec{k}_1 - \vec{k}_2),$$

Why need quantization?

$$\langle \mathscr{R}_{k_1} \mathscr{R}_{k_2}^* \rangle = \langle \hat{v}_{k_1} \hat{v}_{k_2}^* \rangle / z^2 = \left| \frac{v_{k_1}}{z} \right|^2 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \qquad |\mathscr{R}_k| = \left| \frac{v_k}{z} \right|$$

$$= (2\pi^2/k^3) \delta^3(\vec{k}_1 - \vec{k}_2) \mathscr{P}_{\mathscr{R}}(k_1)$$

No knowledge about initial conditions, consider statistical property

Parameterization

$$\mathscr{P}_{\mathscr{R}} = \frac{k^3}{2\pi^2} |\mathscr{R}_k|^2 = A_{\mathscr{R}}(k_*) \left(\frac{k}{k_*} \right)^{n_s - 1 + \frac{1}{2} n_s' \ln(k/k_*) + \cdots}$$

Red tilt $n_s - 1 < 0$

Blue tilt $n_s - 1 > 0$

Spectral tilt

- The power spectrum

$$\mathscr{P}_{\mathscr{R}} = 2^{2\nu-3}\left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right)^2\left(\frac{H}{\dot\phi_0}\right)^2\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}\Bigg|_{k=aH}.$$

$$\mathscr{P}_{\mathscr{R}} \approx [1+2(2-\ln 2-\gamma)(2\epsilon_H-\eta_H)-2\epsilon_H]\left(\frac{H}{\dot\phi_0}\right)^2\left(\frac{H}{2\pi}\right)^2$$

- Spectral index

$$\nu = 3/2 + 2\epsilon_H - \eta_H$$

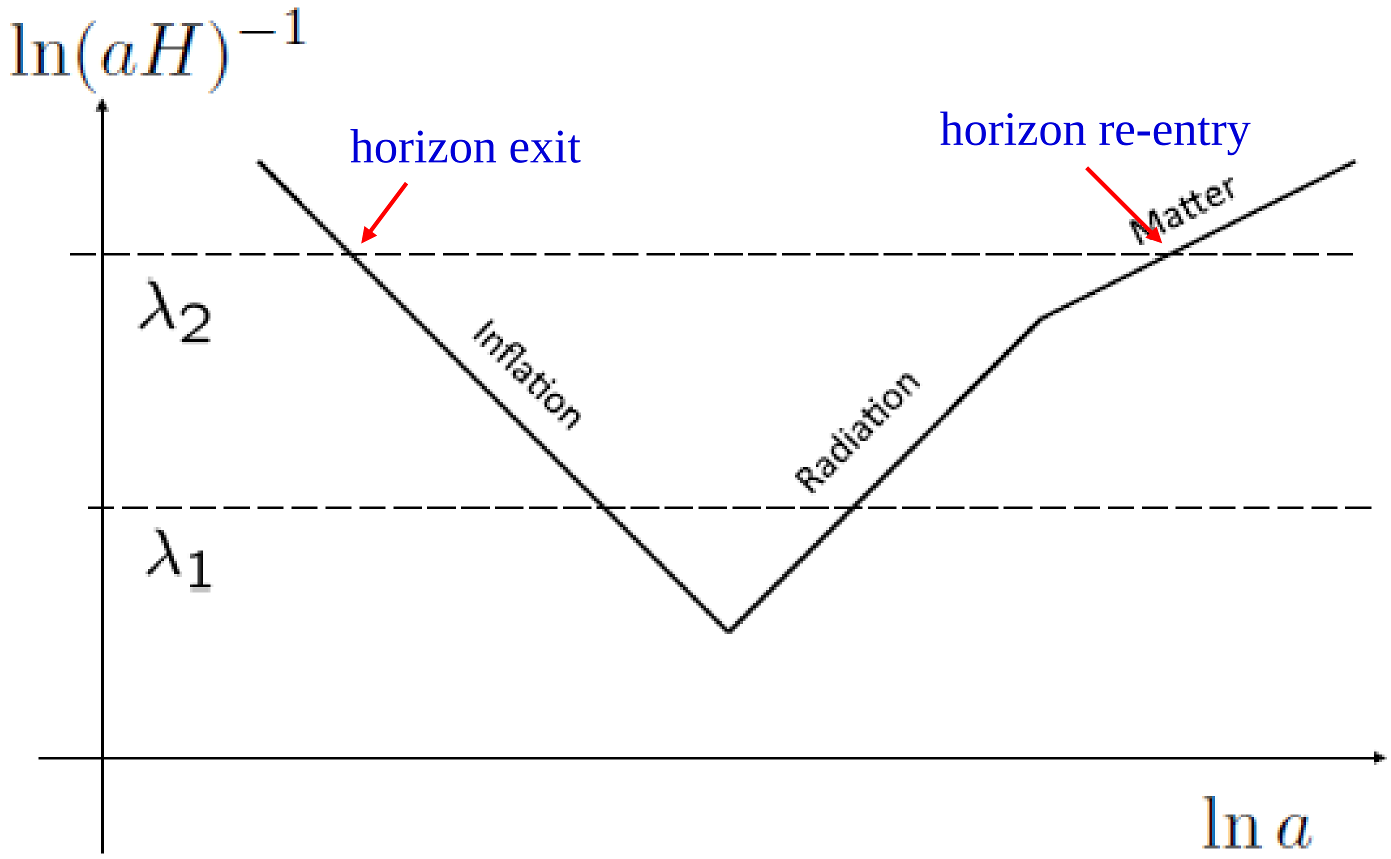
$$n_s - 1 = \frac{d\ln \mathscr{P}_{\mathscr{R}}}{d\ln k}\Bigg|_{k=aH} = 2\eta_H - 4\epsilon_H \approx 2\eta - 6\epsilon$$

$$d\ln k = (1-\epsilon_H)H\,dt$$

$$n_s' = \frac{dn_s}{d\ln k}\Bigg|_{k=aH} = 10\epsilon_H\eta_H - 8\epsilon_H^2 - 2\xi_H$$

$$\dot\epsilon_H = 2H\epsilon_H(\epsilon_H-\eta_H) \qquad \dot\eta_H = H(\epsilon_H\eta_H - \xi_H)$$

Amplification of Quantum Fluctuation



Quantum fluctuation of GWs

- The action to the second order

$$\delta_2 S = \frac{1}{64\pi G}\int d\tau d^3x[(h'_{ij})^2 - (\partial_l h_{ij})^2]a^2$$

$$\hat{h}_{ij}(x,\tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{s=+,\times} [\epsilon^s_{ij}(k)h^s_k(\tau)a_k e^{i\vec{k}\cdot\vec{x}} + (\epsilon^s_{ij}(k)h^s_k(\tau))^* a^\dagger_k e^{-i\vec{k}\cdot\vec{x}}]$$

$$\epsilon_{ii} = k^i\epsilon_{ij} = 0 \qquad \epsilon^s_{ij}\epsilon^{s'}_{ij} = 2\delta_{ss'}$$

$$u^s_k(\tau) = \frac{a}{\sqrt{16\pi G}} h^s_k(\tau)$$

$$\delta_2 S = \sum_s \frac{1}{2}\int d\tau d^3k \left[\left(\frac{du^s_k}{d\tau}\right)^2 - \left(k^2 - \frac{a''}{a}\right)(u^s_k)^2\right]$$

Quantum fluctuation of GWs

- Mode function

$$\frac{d^2u_k^s}{d\tau^2}+\left(k^2-\frac{a''}{a}\right)u_k^s=\frac{d^2u_k^s}{d\tau^2}+\left(k^2-\frac{\mu^2-1/4}{\tau^2}\right)u_k^s=0$$

$$\mu=3/2+\epsilon_H \qquad a''/a=2a^2H^2-a^2H^2\epsilon_H$$

- Asymptotic condition

$$u_k^s(\tau)=\frac{\sqrt{\pi}}{2}e^{i(\mu+1/2)\pi/2}\sqrt{-\tau}H_\mu^{(1)}(-k\tau)$$

- Perturbations on super-horizon

$$u_k^s(\tau)=e^{i(\mu-1/2)\pi/2}2^{\mu-3/2}\frac{\Gamma(\mu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\tau)^{1/2-\mu}$$

Quantum fluctuation of GWs

- The power spectrum

$$\mathscr{P}_T = \frac{k^3}{\pi^2} \sum_{s=+,\times} \left| \frac{2\sqrt{8\pi G}\, u_k^s}{a} \right|^2 = A_T(k_*) \left(\frac{k}{k_*} \right)^{n_T + \frac{1}{2} n'_T \ln(k/k_*) + \cdots}$$

$$= (64\pi G) 2^{2\mu-3} \left(\frac{\Gamma(\mu)}{\Gamma(3/2)} \right)^2 \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{3-2\mu} .$$

$$\mu = 3/2 + \epsilon_H$$

$$\mathscr{P}_T \approx 64\pi G [1 + (1 - \ln 2 - \gamma)\epsilon_H] \left(\frac{H}{2\pi} \right)^2$$

$$\mathscr{P}_T = A_T(k_*) \left(\frac{k}{k_*} \right)^{n_t + \frac{1}{2} n'_t \ln(k/k_*) + \cdots}$$

The tensor spectral tilt

- The spectral index of tensor mode

$$n_T = \frac{d \ln \mathscr{P}_T}{d \ln k} = 3 - 2\mu = -2\epsilon_H$$

- The tensor to scalar ratio

$$\mathscr{P}_{\mathscr{R}} \approx [1 + 2(2 - \ln 2 - \gamma)(2\epsilon_H - \eta_H) - 2\epsilon_H] \left(\frac{H}{\dot{\phi}_0}\right)^2 \left(\frac{H}{2\pi}\right)^2$$

$$\mathscr{P}_T \approx 64\pi G[1 + (1 - \ln 2 - \gamma)\epsilon_H] \left(\frac{H}{2\pi}\right)^2$$

$$r = \frac{\mathscr{P}_T}{\mathscr{P}_{\mathscr{R}}} = 16\epsilon_H = 16\epsilon = -8n_T \qquad \epsilon_H = -\frac{\dot{H}}{H^2} = 4\pi G \left(\frac{\dot{\phi}_0}{H}\right)^2$$

The power spectrum

- The power spectrum is parameterized

$$\mathscr{P}_{\mathscr{R}} = \frac{k^3}{2\pi^2}|\mathscr{R}_k|^2 = A_{\mathscr{R}}(k_*)\left(\frac{k}{k_*}\right)^{n_s-1+\frac{1}{2}n_s'\ln(k/k_*)+\cdots}$$

order of 10^{-9}

$$\mathscr{P}_T = A_T(k_*)\left(\frac{k}{k_*}\right)^{n_t+\frac{1}{2}n_t'\ln(k/k_*)+\cdots} \approx 64\pi G\left(\frac{H}{2\pi}\right)^2$$

$$n_s - 1 = \left.\frac{d\ln\mathscr{P}_{\mathscr{R}}}{d\ln k}\right|_{k=aH} = 3 - 2\nu = 2\eta_H - 4\epsilon_H \approx 2\eta - 6\epsilon$$

$$r = \frac{A_T}{A_{\mathscr{R}}} = 16\epsilon_H = 16\epsilon = -8n_T \qquad A_T = rA_{\mathscr{R}} \sim H^2 \sim V(\phi)$$

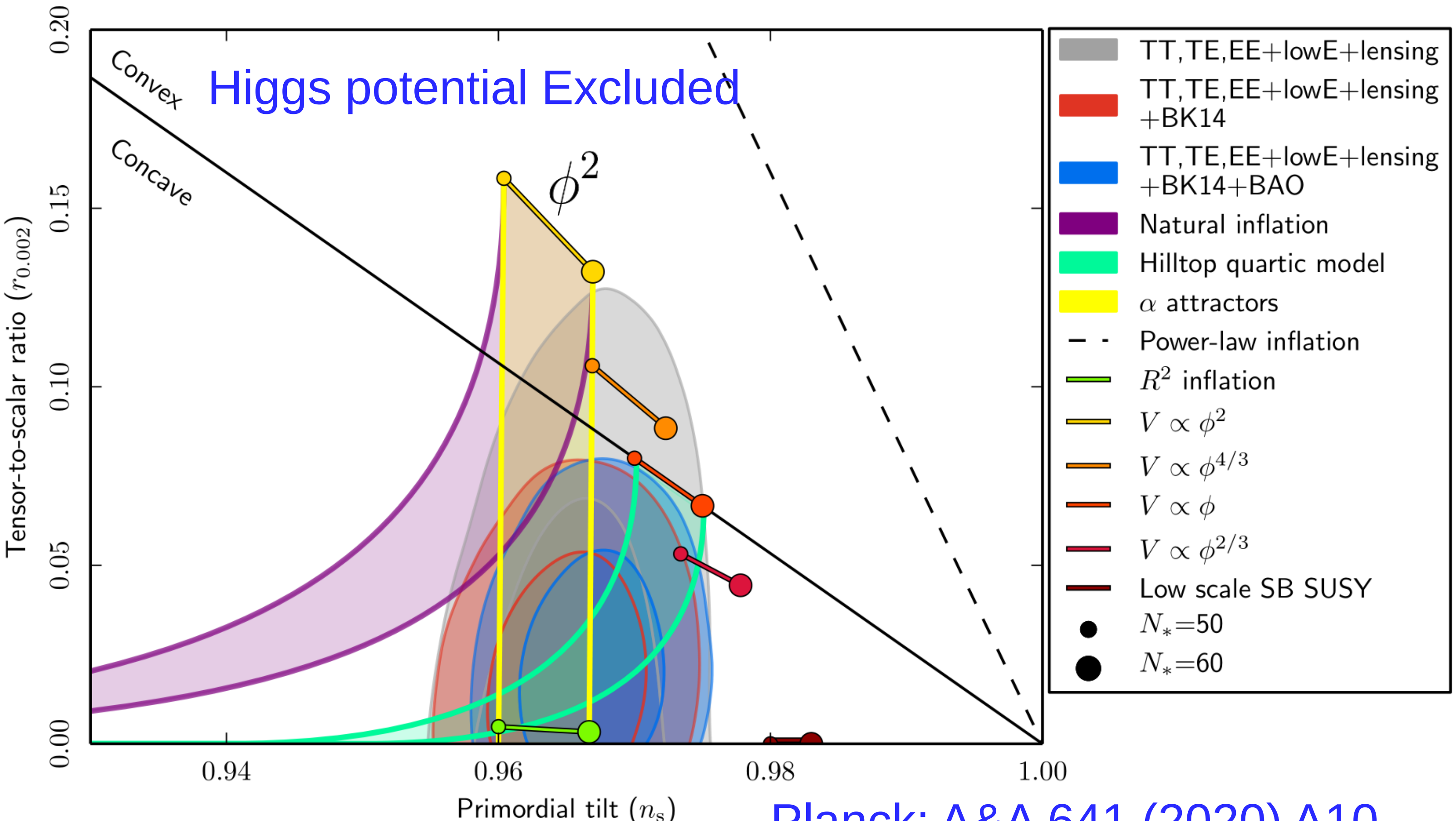
- Energy scale of inflation: measurement of r

CMB constraints

- Planck 18

$$\ln(10^{10}A_s) = 3.044 \pm 0.014$$ order of 10^{-9}

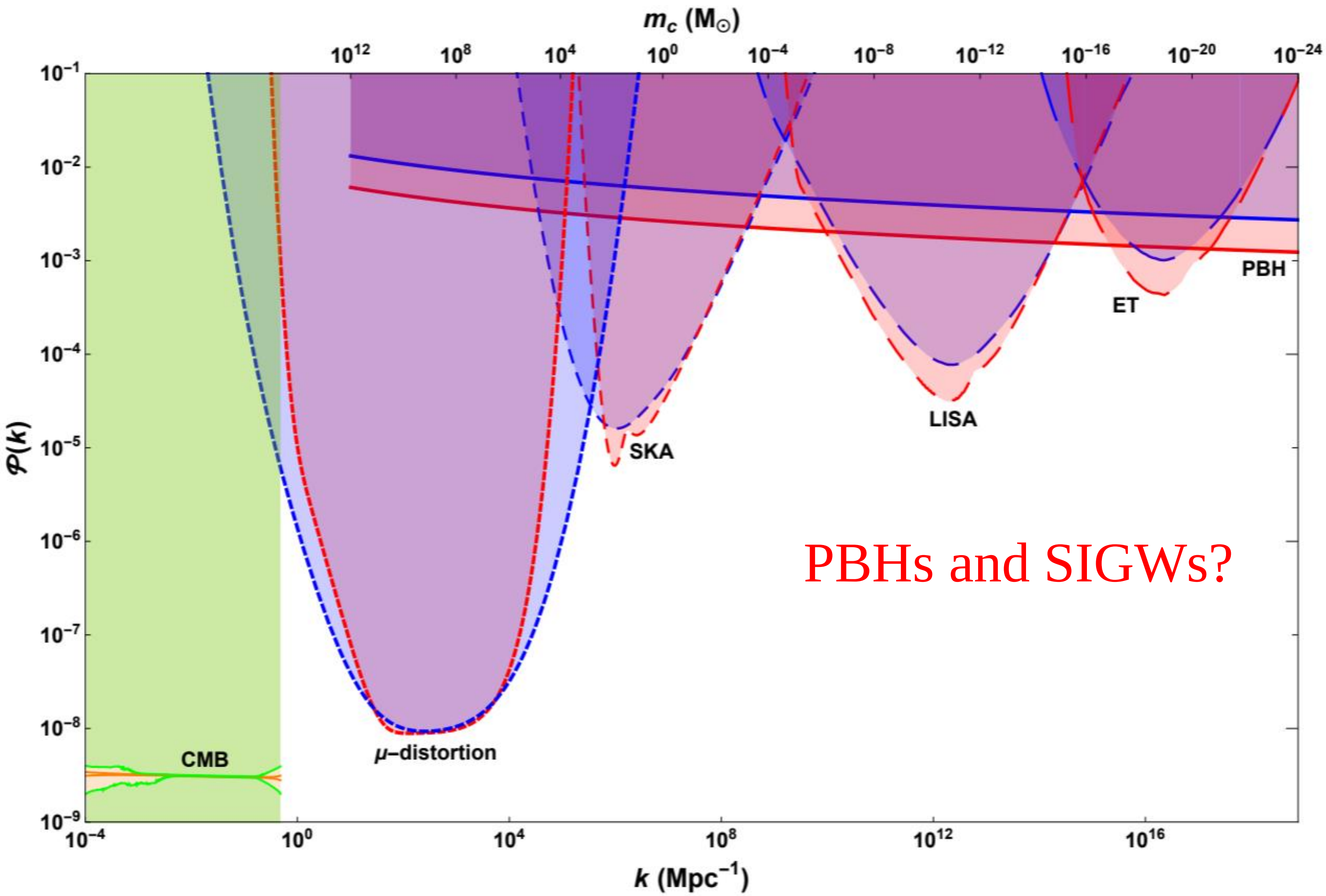
$$r_{0.05} < 0.06, \quad 95\%$$



Planck: A&A 641 (2020) A10

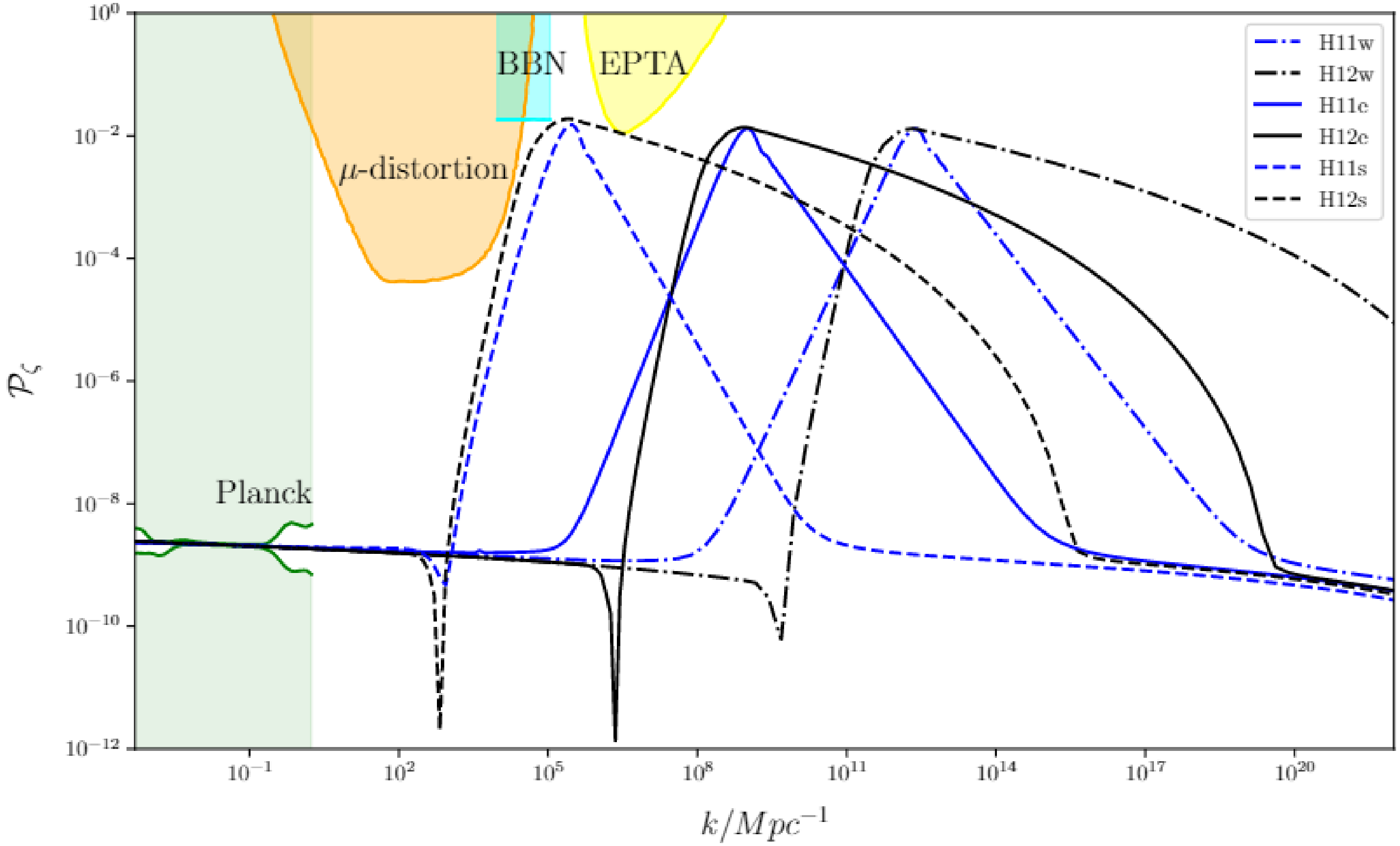
BICEP: PRL 121 (2018) 221301

GW Constraints



Gow, Byrnes, Cole, Young, JCAP 02 (2021) 002

Enhancement models



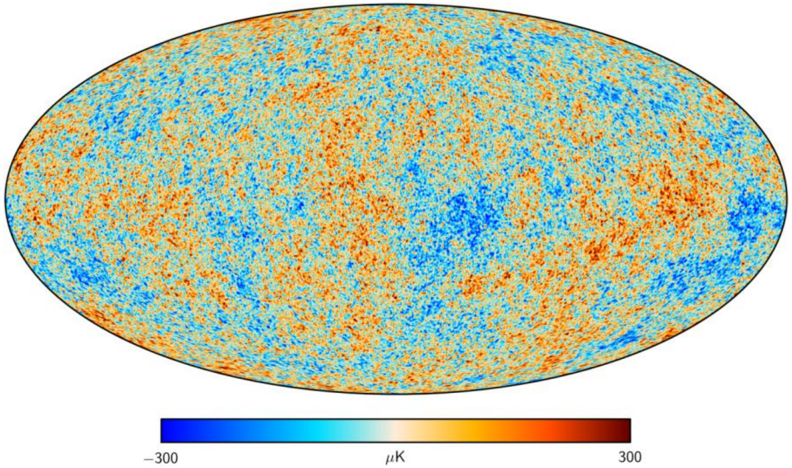
Di and Gong, 1707.09578 (JCAP); Lin etal, 2001.05909 (PRD); Yi etal., 2007.09957 (PRD); Yi etal., 2011.10606 (PRD); Zhang etal., 2012.06960 (JCAP); Gao etal., 2012.03856 (NPB); Gao etal., 2102.07369 (SCMPA); Wu etal., 2105.07694

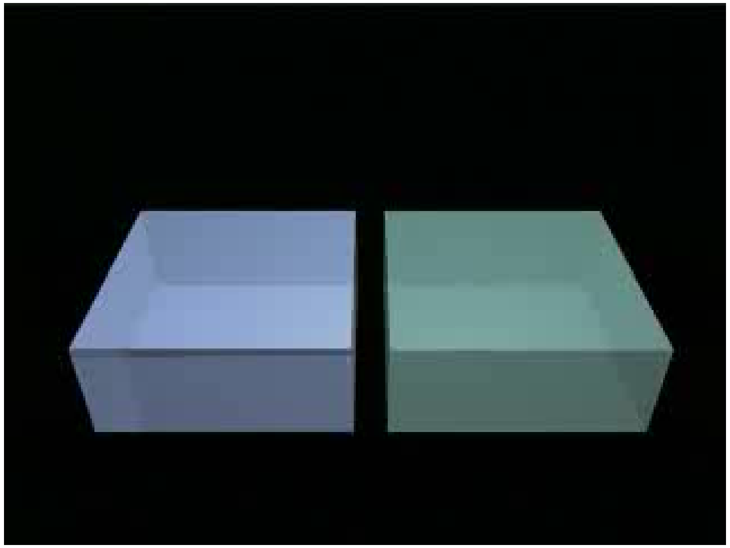
Summary of Inflation

- At $t=10^{-32}$ s, the Universe is about 10^{-24}cm
- In about 10^{-33}s, the Universe expanded exponentially by a factor of 10^{30}
- The quantum fluctuation of the inflaton seeds the formation of the large scale structure, and leaves imprints as small anisotropy in CMB (COBE in 1991)
- The power spectrum of the density perturbation is almost Gaussian, adiabatic, and scale invariant

Inflation

- Ripples: The explosive expansion of space during inflation would have created ripples in the fabric of space.
- GW: these gravity waves should have left a signature in the polarization of the last-scattered photons (CMB).





Reheating

- After the end of inflation, all the energy of the universe is stored in the inflaton, and the temperature is extremely low.
- A process of energy transfer is needed to keep thermal equilibrium, and recovers the standard thermal history.

$$\dot{\rho} + (3H + \Gamma)\rho = 0$$

Particle decay rate Γ

$$T_{reh} \sim \sqrt{M_{pl}\Gamma}$$

$$\frac{T_{reh}}{1\ \mathrm{GeV}} \sim \left(\frac{m}{10^6\ \mathrm{GeV}}\right)^{3/2} \qquad m \gtrsim 10^4\ \mathrm{GeV}$$

Problems

- Too many models: which model of inflation is correct?

  chaotic inflation, Higgs inflation, natural inflation
  Hilltop, Spontaneously broken SUSY, Hybrid
  DBI, D-brane, racetrack, R^2 inflation, α attractors,
  ...

- Why did inflation happen? what is the initial condition?
- What about other parts of inflating universe? We only see a small part, what is the rest?

CMB

- CMB arises from last-scattering surface ~300,000 years after the Big Bang

Electrons and protons recombined, the opacity of the universe to Thomson scattering went to zero, and the photons were then free to stream to us.

- This is the earliest direct image of the Universe we can ever obtain (EM anyway…)
- The imprints of structure of the Universe today AND BigBang/inflation should also be imprinted there...

CMB anisotropy

- Ripples in the space-time causes density fluctuations
- The slightly over-dense regions later became gravitationally unstable and collapsed to form galaxies, clusters of galaxies and all other structures we see in the Universe today
- over-dense regions recombined first
- The denser regions cause the CMB photons to be gravitationally redshifted compared to photons arising in less dense regions (climbing out the potential wells).

Sources of CMB Anisotropy

- Photon density perturbations: Over-densities of photons look hotter $\rho_r \sim T^4$
- Doppler shift: Velocity of photon/baryons at last scattering gives Doppler shift
- Sachs-Wolfe effect: The amplitude of the temperature fluctuations is roughly 1/3 of the density fluctuations.
- Integrated Sachs-Wolfe effect: Evolution of potential along photon line of sight causes net red- or blue-shift as photons climbs in and out of time-varying potential wells

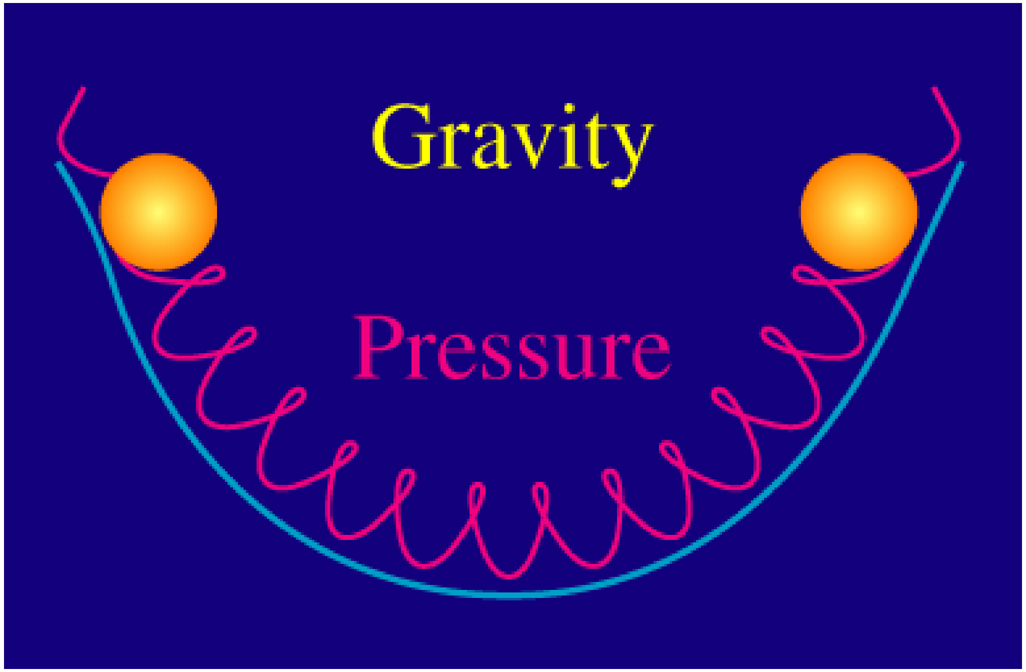
Photon-baryon plasma

- Plasma: At t=300,000 years, the universe consisted of a plasma of mainly electrons, protons, and CMB photons. Because there was also a small amount of helium and heavier elements which contain neutrons, we usually refer to the mix as a photon-baryon plasma
- Before recombination, free electrons act to glue the CMB photons to the baryons by Thomson scatting, the plasma behaves as a nearly perfect fluid

Sound Waves

- Acoustic oscillations: Gravity tries to compress the fluid in potential wells, radiation pressure resists any attempt to compress the fluid setting up acoustic oscillations

we will represent the radiation pressure abstractly as springs. Likewise we will represent the inertia of the fluid, or loosely speaking its mass (really energy density), as massive balls falling under gravity:



Acoustic Oscillations

- We don't actually hear the sound of these acoustic waves. What we actually see is the pattern of the sound waves that is imprinted on the temperature of the CMB.
- Compressing a gas heats it up. Letting it expand cools it down. The CMB is locally hotter in regions where the acoustic wave causes compression and cooler where it causes rarefaction

Harmonics

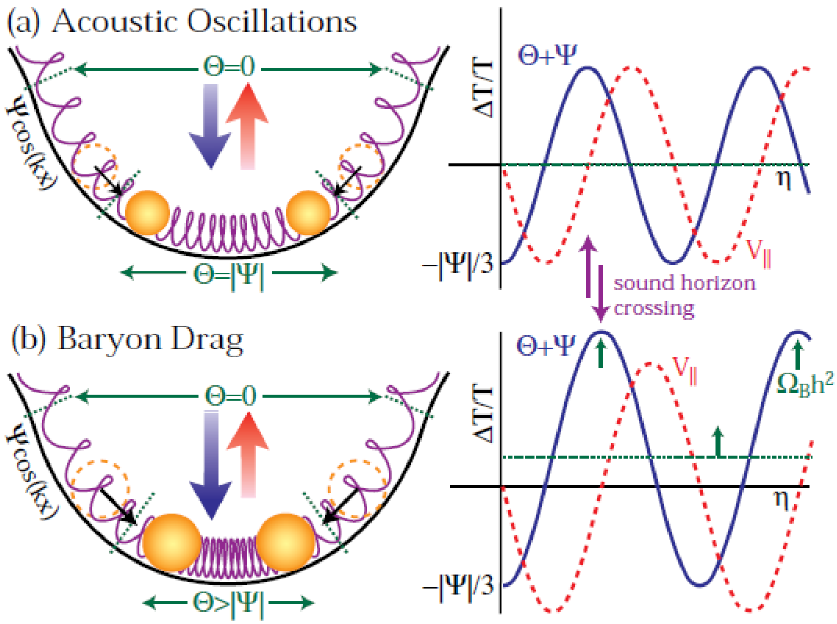
- Frozen: Sound waves stop oscillating at recombination when the baryons release the photons. Modes that reach extrema of their oscillation (maximal compression or rarefaction in potential wells) by recombination will carry enhanced temperature fluctuations
- Peaks: There is a special mode for which the fluid just has enough time to compress once before recombination, Mathematically the wavenumber (2π/wavelength) of this mode (k_1) is equal to π divided by the distance sound can travel by recombination
- Sound horizon: This distance is called the sound horizon at recombination.

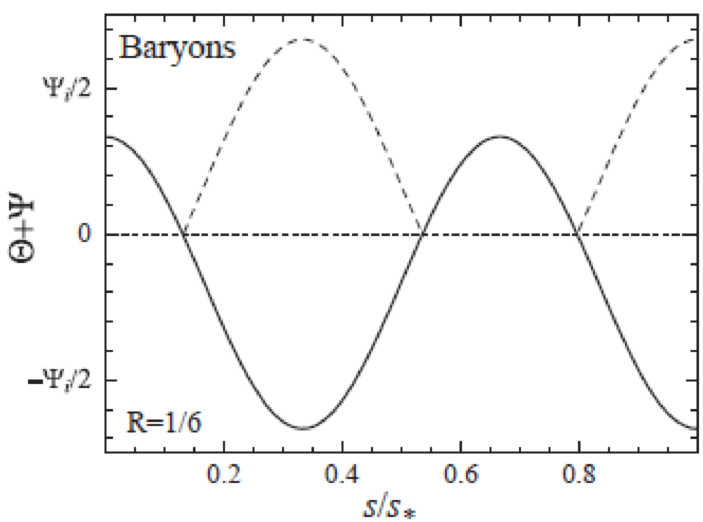
Acoustic Oscillation

- Harmonics and Peaks: Modes caught at extrema of their oscillations become the peaks in the CMB power spectrum. They form a harmonic series based on the sound horizon. The first peak represents the mode that compressed once inside potential wells before recombination, the second the mode that compressed and then rarefied, the third the mode that compressed then rarefied then compressed

Baryon Loading

- Baryons: Add inertia to the fluid
- Equivalent to add mass in a spring
- Lead to the shift of the equilibrium point of the spring
- Un-equal amplitudes of extrema or peaks





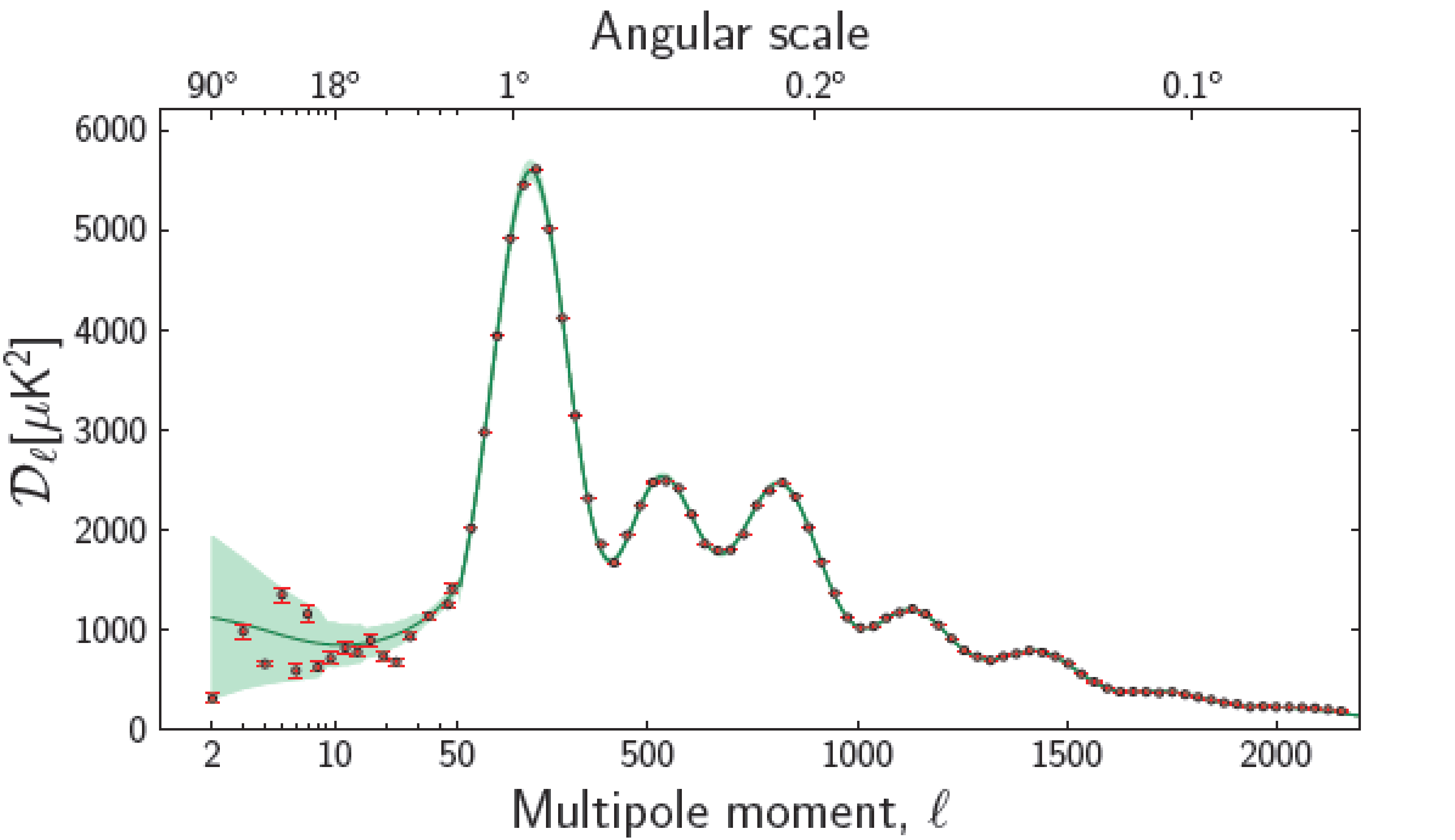
Silk damping

- Random work:

Remember that recombination does not occur instantaneously. In that short period during which the universe recombines, the photons bounce around the baryons and execute a random walk, if the physical scale of these fluctuations are so small that they are comparable to the distance photons travel during recombination, then the hot and cold photons mix and average out. The acoustic oscillations are exponentially damped on scales smaller than the distance photons random walk during recombination.

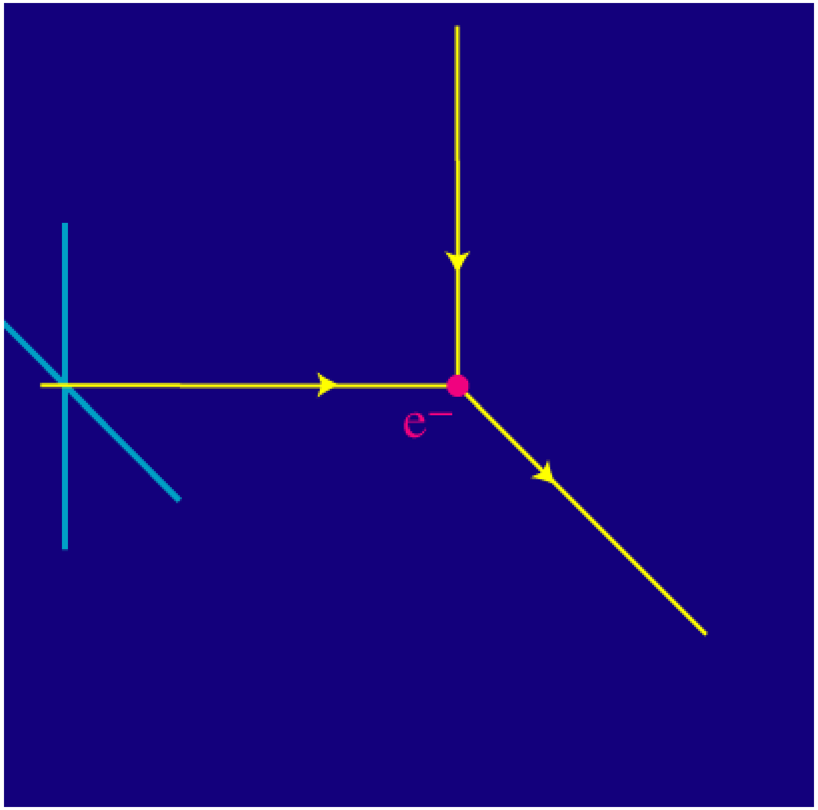
- On larger scales, photons are tightly coupled with electrons which are in turn tightly coupled with baryons (no anisotropy stress in the fluid)

Power Spectrum



Polarization by scattering

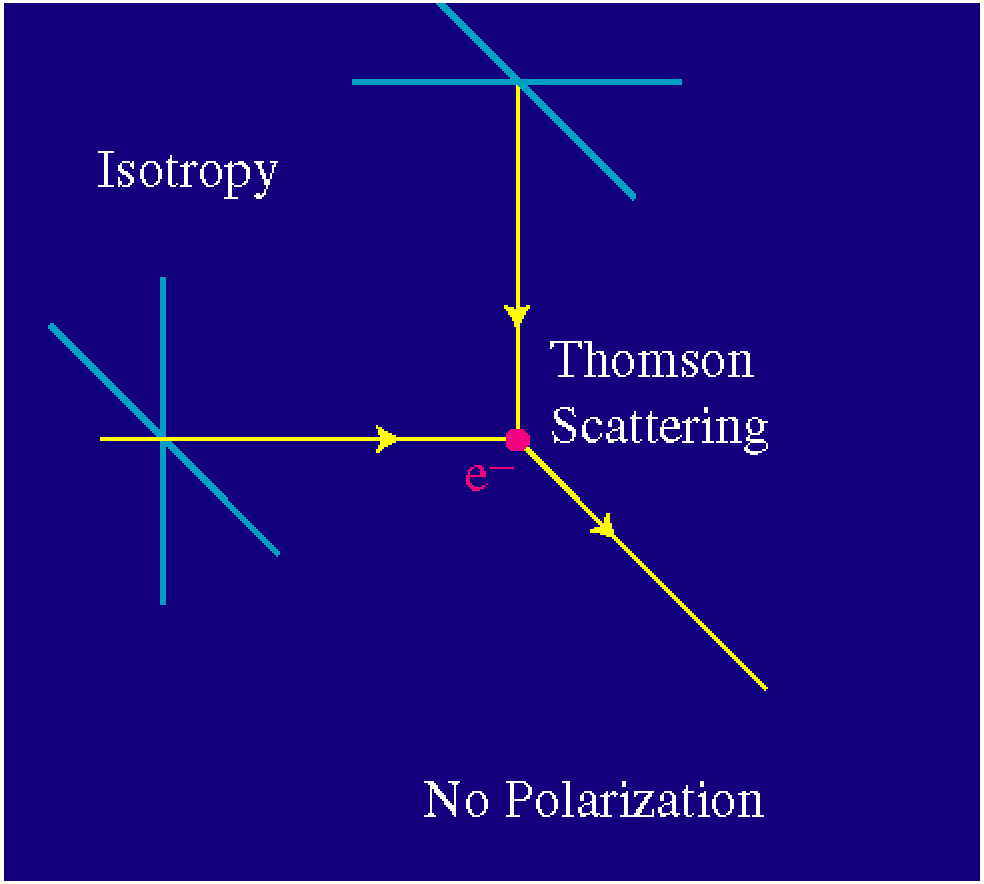
- Thomson scatting



When a light is incident on a free electron, Since light cannot be polarized along its direction of motion, only one linear polarization state gets scattered, the scattered wave is polarized perpendicular to the incidence direction.

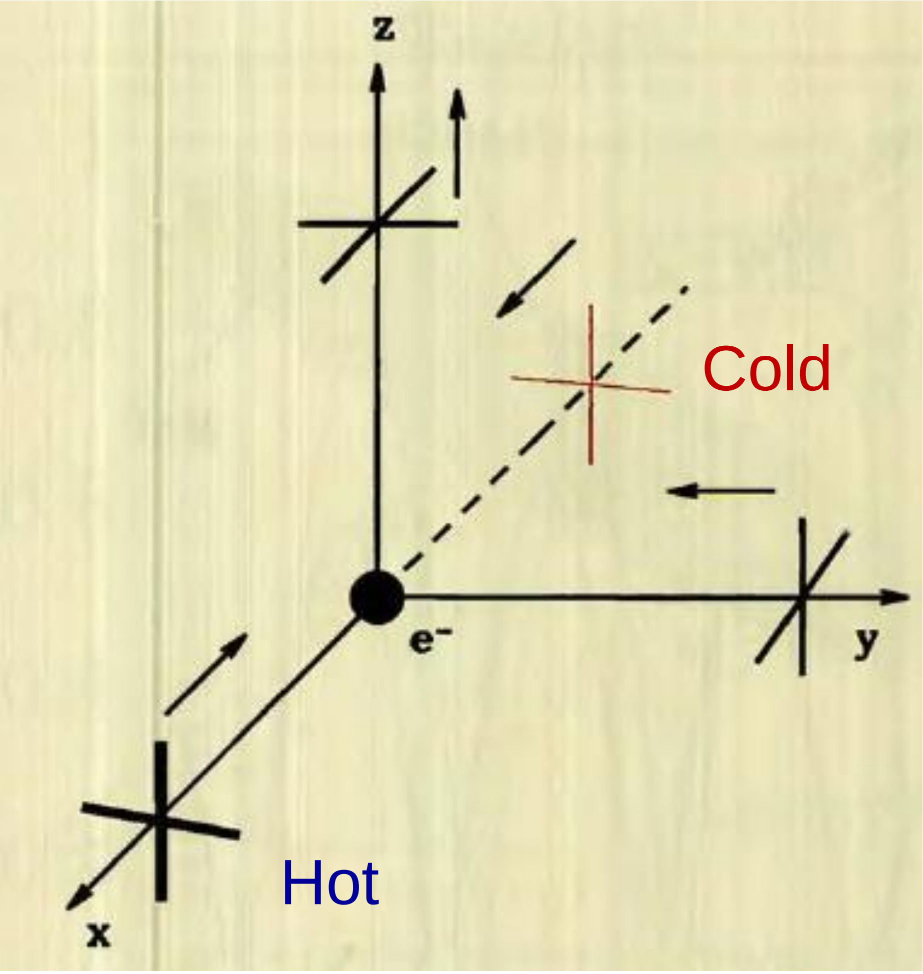
Thomson Scattering

- Isotropic light: no polarization

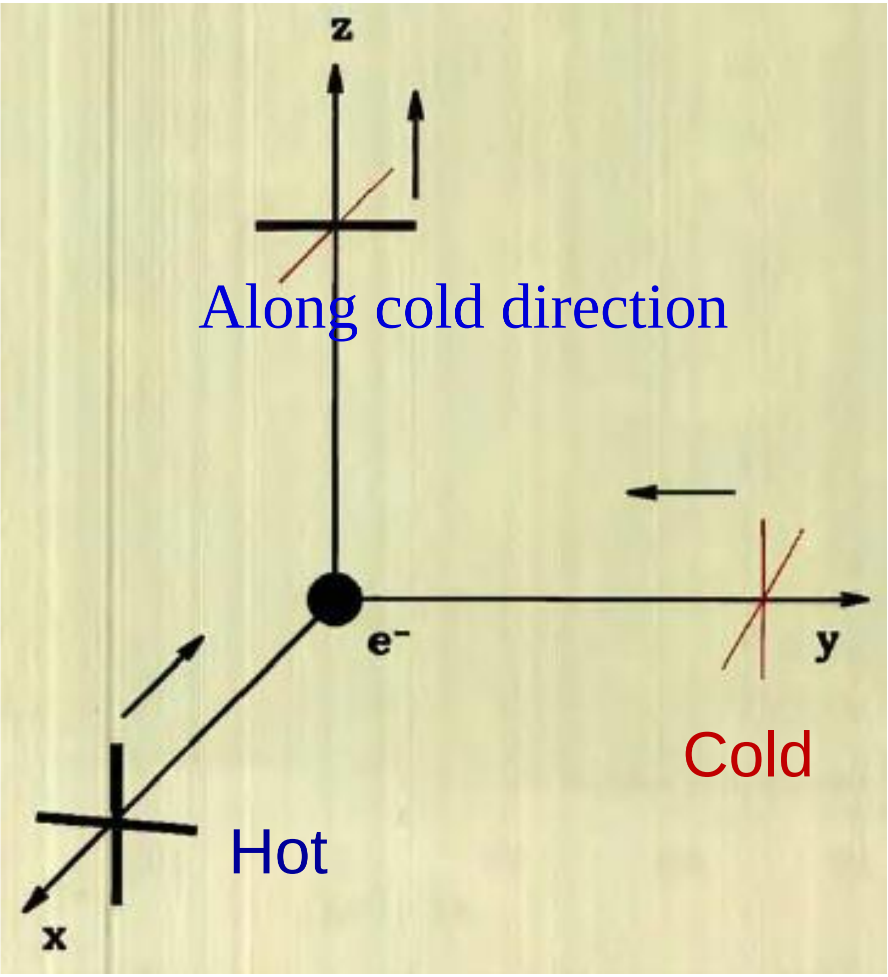


Of course there is nothing particularly special about light coming in from the left. Consider instead light coming in from the top, Now the outgoing radiation possesses both polarization states. If the incoming radiation from the left and top are of equal intensity, the result is no polarization in the outgoing direction.

Thomson scattering

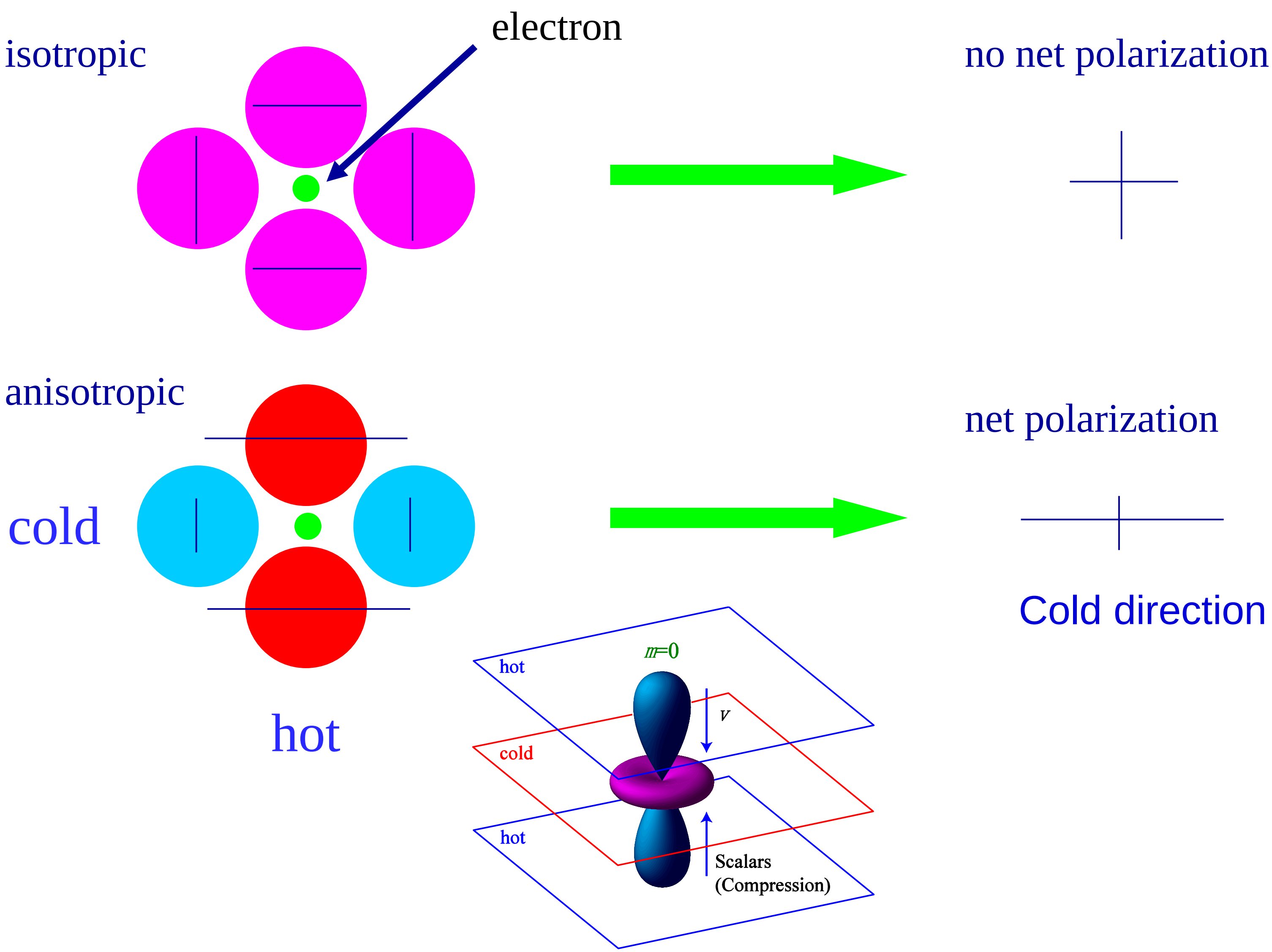


Dipole

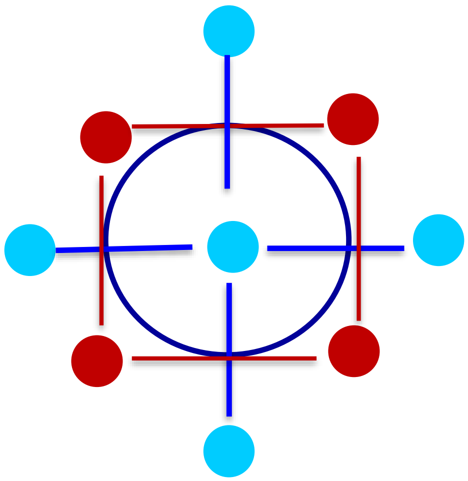


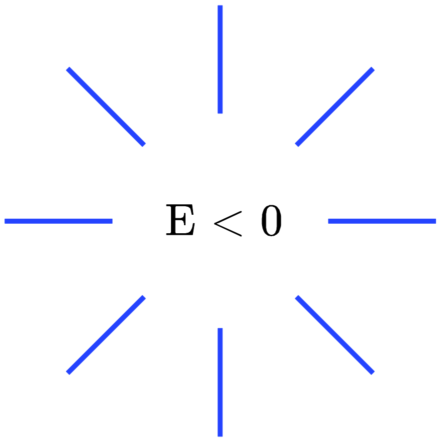
Quadrupole

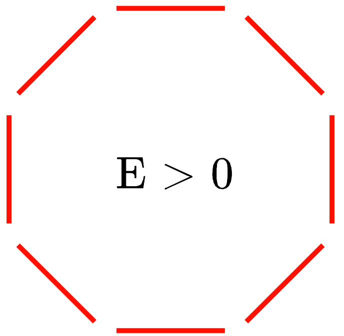
Polarization by Quadrupole Anisotropy

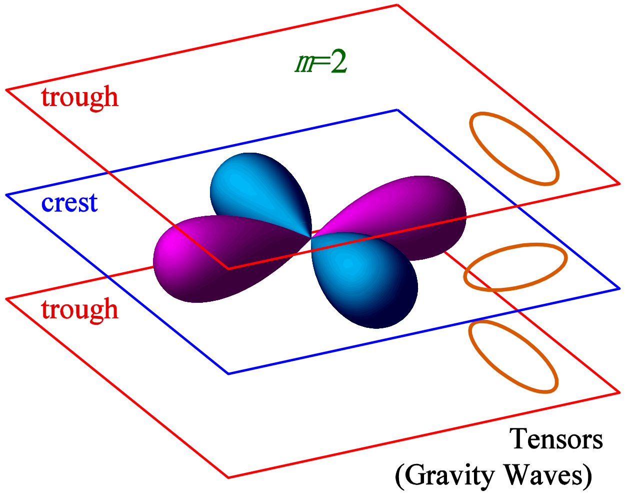


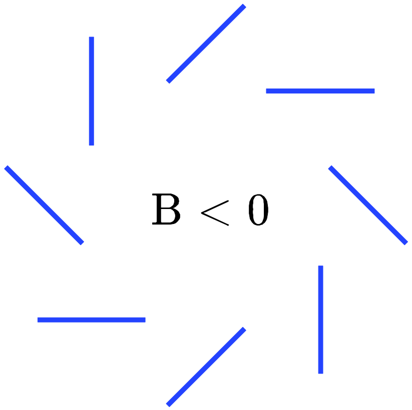
E mode and B mode

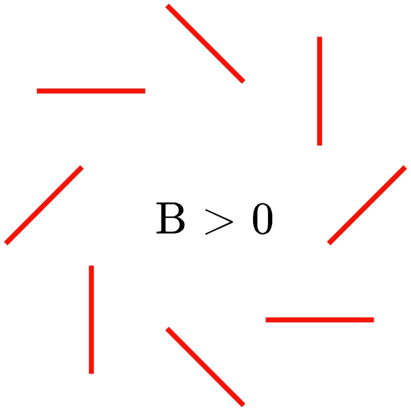




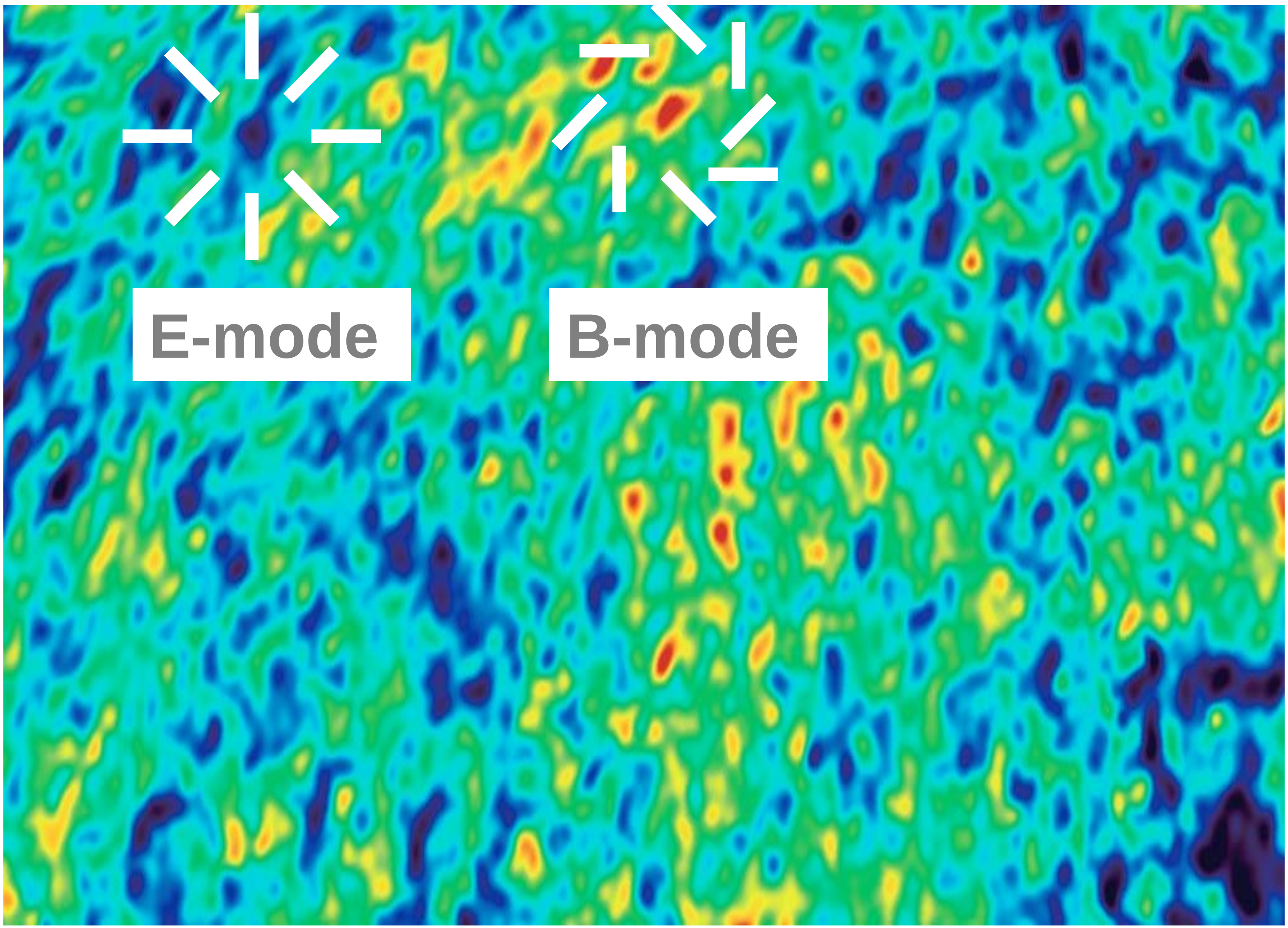








E and B modes in CMB



E模及B模

- 对称无迹张量分解

  梯度分量 $E_{;ab}-(1/2)g_{ab}E_{;c}^{\ \ c}$

  旋度分量 $(B_{;ac}\epsilon^{c}_{\ b}+B_{;bc}\epsilon^{c}_{\ b})/2$

- 极化张量分解：E模

$$Q_E=\frac{1}{2}\left(\partial_\theta^2-\cot\theta\partial_\theta-\csc^2\theta\partial_\phi^2\right)E,$$

$$U_E=\csc\theta\left(\partial_\theta\partial_\phi-\cot\theta\partial_\phi\right)E.$$

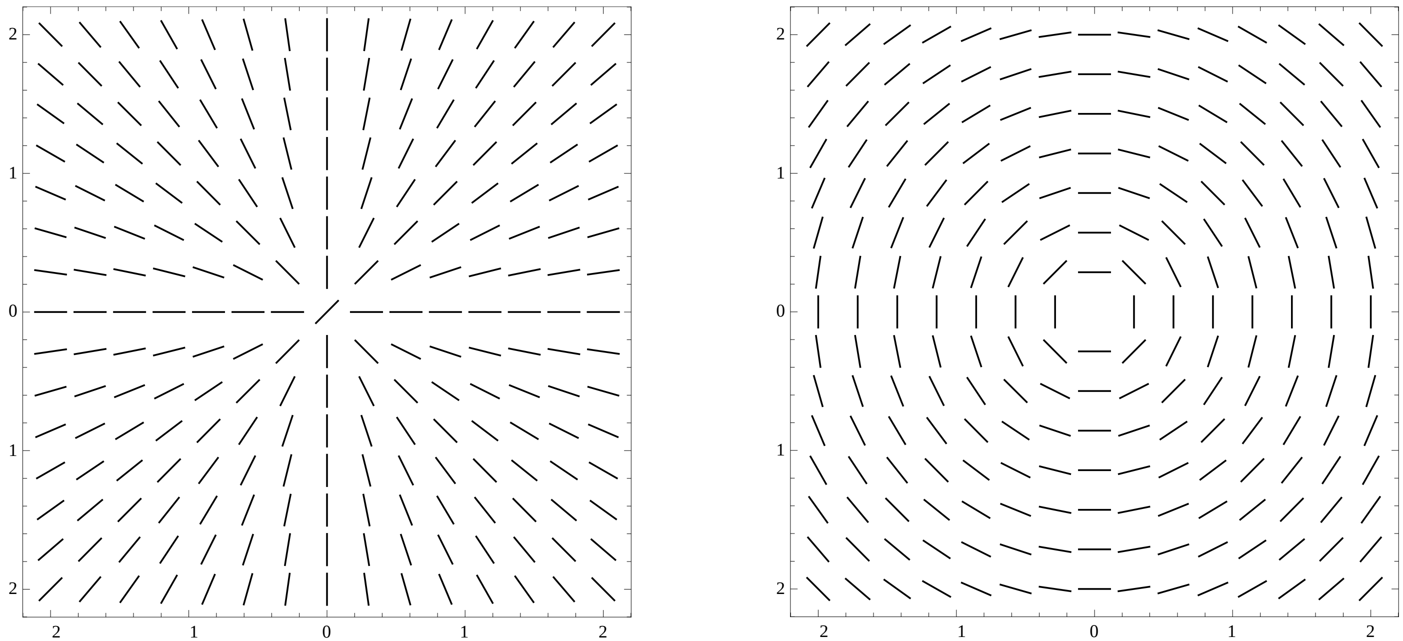
二维平直空间（小尺度近似）

$$Q_E=\frac{1}{2}(\partial_x^2-\partial_y^2)E,\quad U_E=\partial_x\partial_yE,$$

E模

- E模极化图

$$\sqrt{Q^2+U^2}, \quad \alpha = \frac{1}{2}\arctan(U/Q)$$



转动变化下不变

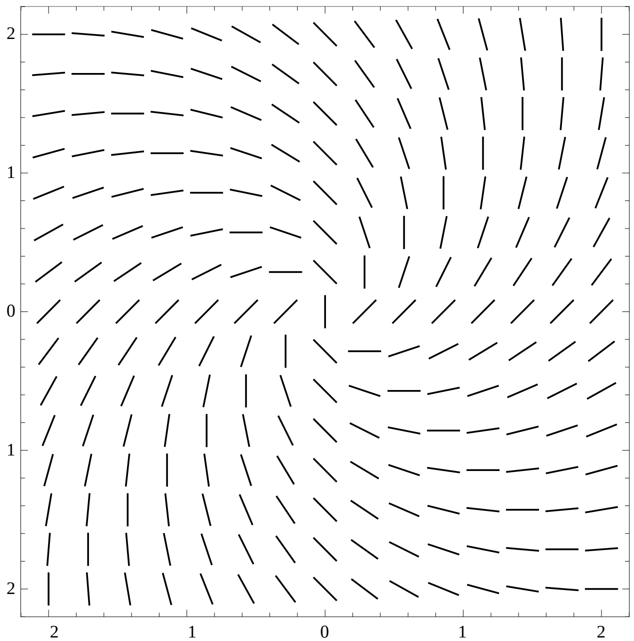
极化张量分解

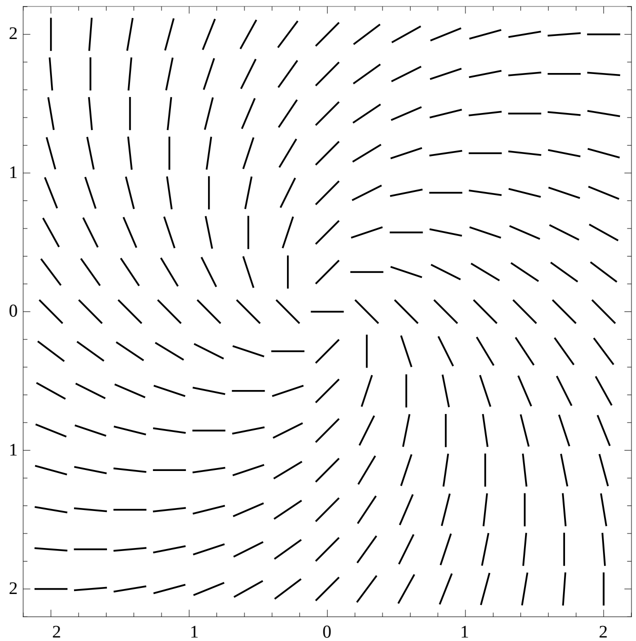
- B模分量

$$Q_B = -\csc\theta\left(\partial_\theta\partial_\phi - \cot\theta\partial_\phi\right)B,$$

$$U_B = \frac{1}{2}\left(\partial_\theta^2 - \cot\theta\partial_\theta - \csc^2\theta\partial_\phi^2\right)B.$$

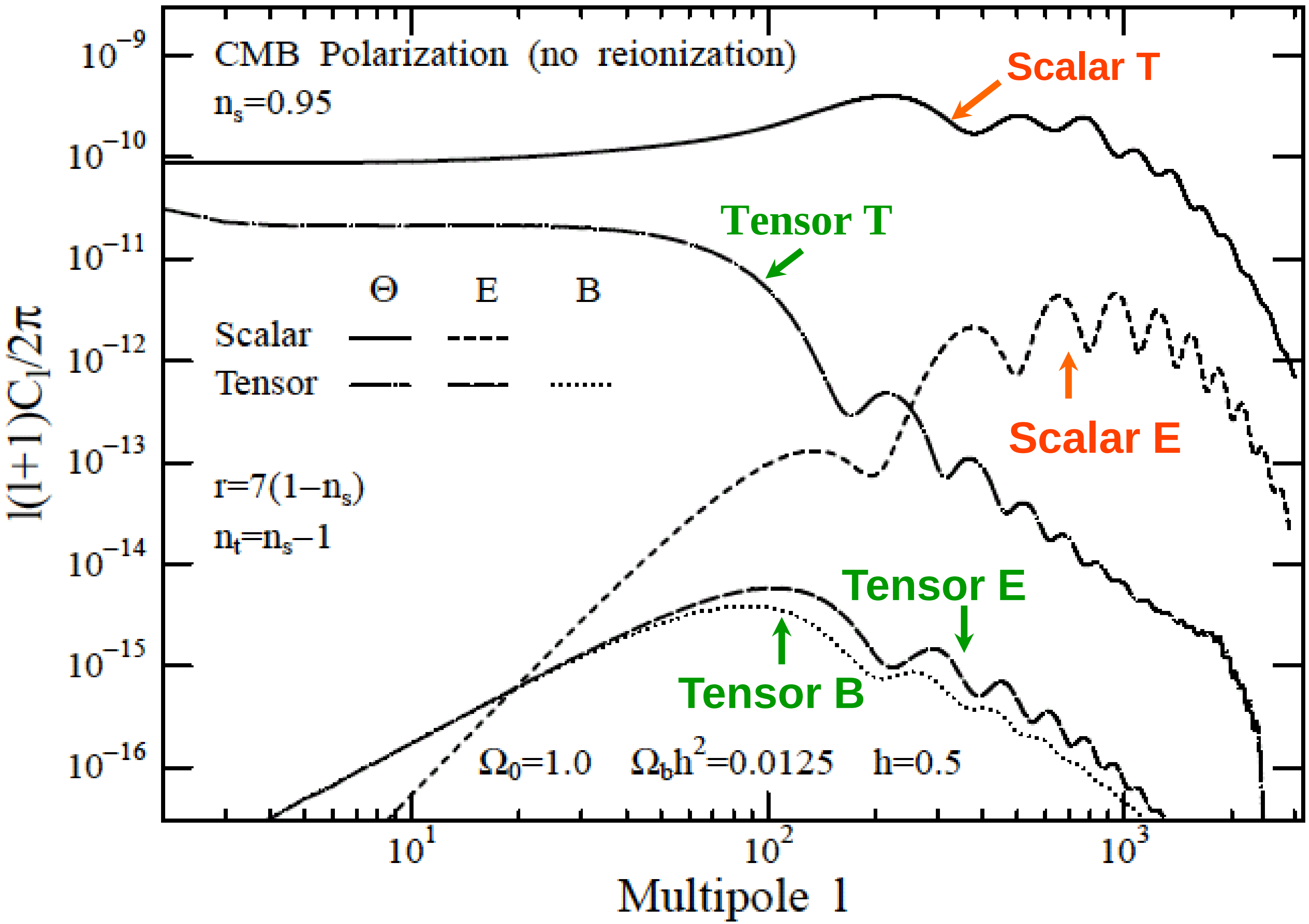
二维平直空间 $Q_B = -\partial_x\partial_y B, \quad U_B = \frac{1}{2}(\partial_x^2 - \partial_y^2)B$





转动变化下不变

Power Spectrum



Cosmological parameters

(Planck 2018, 1807.06209, *AA* 641 (2020) A6)

Parameter	TT+lowE 68% limits	TE+lowE 68% limits	EE+lowE 68% limits	TT,TE,EE+lowE 68% limits	TT,TE,EE+lowE+lensing 68% limits	TT,TE,EE+lowE+lensing+BAO 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02249 ± 0.00025	0.0240 ± 0.0012	0.02236 ± 0.00015	0.02237 ± 0.00015	0.02242 ± 0.00014
$\Omega_c h^2$	0.1206 ± 0.0021	0.1177 ± 0.0020	0.1158 ± 0.0046	0.1202 ± 0.0014	0.1200 ± 0.0012	0.11933 ± 0.00091
$100\theta_{MC}$	1.04077 ± 0.00047	1.04139 ± 0.00049	1.03999 ± 0.00089	1.04090 ± 0.00031	1.04092 ± 0.00031	1.04101 ± 0.00029
τ	0.0522 ± 0.0080	0.0496 ± 0.0085	0.0527 ± 0.0090	$0.0544^{+0.0070}_{-0.0081}$	0.0544 ± 0.0073	0.0561 ± 0.0071
$\ln(10^{10} A_s)$	3.040 ± 0.016	$3.018^{+0.020}_{-0.018}$	3.052 ± 0.022	3.045 ± 0.016	3.044 ± 0.014	3.047 ± 0.014
n_s	0.9626 ± 0.0057	0.967 ± 0.011	0.980 ± 0.015	0.9649 ± 0.0044	0.9649 ± 0.0042	0.9665 ± 0.0038
H_0 [km s^{-1} Mpc^{-1}]	66.88 ± 0.92	68.44 ± 0.91	69.9 ± 2.7	67.27 ± 0.60	67.36 ± 0.54	67.66 ± 0.42
Ω_Λ	0.679 ± 0.013	0.699 ± 0.012	$0.711^{+0.033}_{-0.026}$	0.6834 ± 0.0084	0.6847 ± 0.0073	0.6889 ± 0.0056
Ω_m	0.321 ± 0.013	0.301 ± 0.012	$0.289^{+0.026}_{-0.033}$	0.3166 ± 0.0084	0.3153 ± 0.0073	0.3111 ± 0.0056
$\Omega_m h^2$	0.1434 ± 0.0020	0.1408 ± 0.0019	$0.1404^{+0.0034}_{-0.0039}$	0.1432 ± 0.0013	0.1430 ± 0.0011	0.14240 ± 0.00087
$\Omega_m h^3$	0.09589 ± 0.00046	0.09635 ± 0.00051	$0.0981^{+0.0016}_{-0.0018}$	0.09633 ± 0.00029	0.09633 ± 0.00030	0.09635 ± 0.00030
σ_8	0.8118 ± 0.0089	0.793 ± 0.011	0.796 ± 0.018	0.8120 ± 0.0073	0.8111 ± 0.0060	0.8102 ± 0.0060
$S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$	0.840 ± 0.024	0.794 ± 0.024	$0.781^{+0.052}_{-0.060}$	0.834 ± 0.016	0.832 ± 0.013	0.825 ± 0.011
$\sigma_8 \Omega_m^{0.25}$	0.611 ± 0.012	0.587 ± 0.012	0.583 ± 0.027	0.6090 ± 0.0081	0.6078 ± 0.0064	0.6051 ± 0.0058
z_{re}	7.50 ± 0.82	$7.11^{+0.91}_{-0.75}$	$7.10^{+0.87}_{-0.73}$	7.68 ± 0.79	7.67 ± 0.73	7.82 ± 0.71
$10^9 A_s$	2.092 ± 0.034	2.045 ± 0.041	2.116 ± 0.047	$2.101^{+0.031}_{-0.034}$	2.100 ± 0.030	2.105 ± 0.030
$10^9 A_s e^{-2\tau}$	1.884 ± 0.014	1.851 ± 0.018	1.904 ± 0.024	1.884 ± 0.012	1.883 ± 0.011	1.881 ± 0.010
Age [Gyr]	13.830 ± 0.037	13.761 ± 0.038	$13.64^{+0.16}_{-0.14}$	13.800 ± 0.024	13.797 ± 0.023	13.787 ± 0.020
z_*	1090.30 ± 0.41	1089.57 ± 0.42	$1087.8^{+1.6}_{-1.7}$	1089.95 ± 0.27	1089.92 ± 0.25	1089.80 ± 0.21
r_* [Mpc]	144.46 ± 0.48	144.95 ± 0.48	144.29 ± 0.64	144.39 ± 0.30	144.43 ± 0.26	144.57 ± 0.22
$100\theta_*$	1.04097 ± 0.00046	1.04156 ± 0.00049	1.04001 ± 0.00086	1.04109 ± 0.00030	1.04110 ± 0.00031	1.04119 ± 0.00029
z_{drag}	1059.39 ± 0.46	1060.03 ± 0.54	1063.2 ± 2.4	1059.93 ± 0.30	1059.94 ± 0.30	1060.01 ± 0.29

Hubble constant

- Hubble measurement（1929）

$$H_0 = 500 \text{ km/s/Mpc}$$

- Supernovae measurements

$$H_0 = 62.3 \pm 1.3(\text{random}) \pm 4(\text{systematic}) \text{ km/s/Mpc}$$

G. Tammann, A. Sandage, and B. Reindl, AA Rev. 15, 289 (2008)

$$H_0 = 73.8 \pm 2.4 \text{ km/s/Mpc}$$

A. G. Riess et~al., Astrophys. J. 730, 119 (2011)

$$H_0 = 74.3 \pm 2.1(\text{systematic}) \text{ km/s/Mpc}$$

W.L. Freedman et~al., Astrophys. J. 758, 24 (2012)

$$H_0 = 70.6 \pm 3.3 \text{ km/s/Mpc}$$ **Recalibration**

$$H_0 = 72.5 \pm 2.5 \text{ km/s/Mpc}$$ **3 different calibrations**

G. Efstathiou，arXiv: 1311.3461, MNRAS 440 (2014) 1138

Hubble constant

- Supernovae measurements

$$H_0 = 73.24 \pm 1.74 \text{ km/s/Mpc}$$ A.G. Riess etal., ApJ 826 (2016) 56

$$H_0 = 74.03 \pm 1.42 \text{ km/s/Mpc}$$

Calibration: LMC (Large Magellanic Cloud) DEBs (detached eclipsing binaries), masers in NGC 4258, Milky Way parallaxes

$$H_0 = 69.8 \pm 0.6(\text{stat}) \pm 1.6(\text{sys}) \text{ km/s/Mpc}$$

W.L. Freedman, arXiv: 2106.15656

Cepheid Calibration Problem?

$$H_0 = 66.9 \pm 2.5 \text{ km/s/Mpc}$$

$$H_0 = 71.8 \pm 1.6 \text{ km/s/Mpc}$$

Cepheid Calibration, color-luminosity relation

E. Mortsell etal., arXiv: 2105.11461

Hubble constant

- Planck Result (LCDM model)

$$H_0 = 67.27 \pm 0.6\ \mathrm{km/s/Mpc}$$ TT,TE,EE+lowE

Planck 2018, 1807.06209, *AA* 641 (2020) A6

- LIGO/Virgo (GW170817)

$$d_L = \frac{5c^6}{96\pi^2}\frac{1}{2^{1/3}f^{5/2}A(f)}\sqrt{\frac{\dot{f}(t)}{f(t)}} \qquad d_L = 43.8^{+2.9}_{-6.9}\ \mathrm{Mpc}$$

NGC 4993 (Tully-Fisher relation) $d_L = 41.1 \pm 5.8\ \mathrm{Mpc}$

maximum median

$$H_0 = 70^{+12}_{-8}\ \mathrm{km/s/Mpc} \qquad H_0 = 74^{+16}_{-8}\ \mathrm{km/s/Mpc}$$

LIGO/Virgo, Nature 551 (2017) 85

Hubble tension

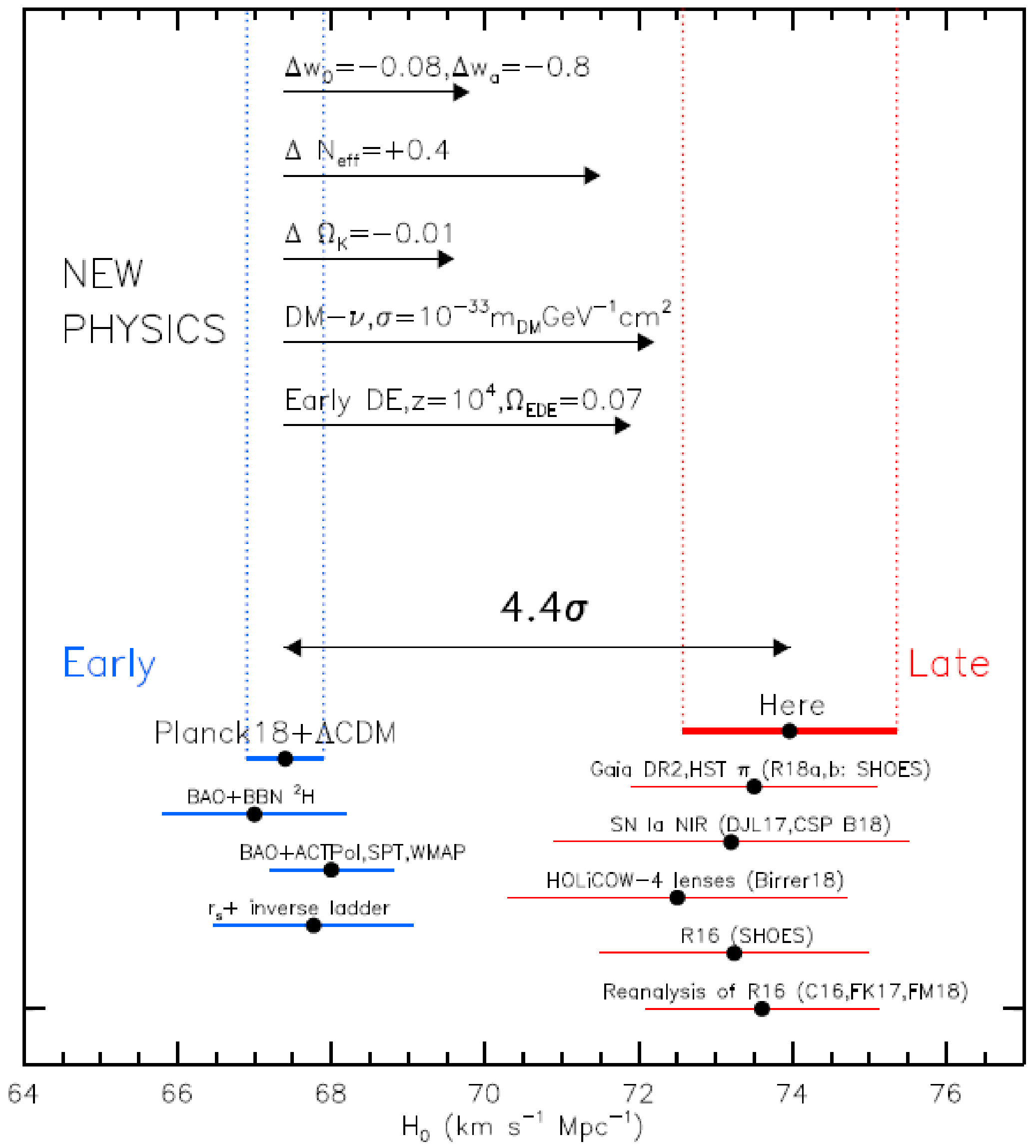
- The tension

Riess et al., ApJ 876 (2019) 85

引力波限制

$$H_0 = 70^{+12}_{-8} \text{ Km s}^{-1} \text{ Mpc}^{-1}$$

LIGO/Virgo, PRL 119 (2017) 161101



Thank You